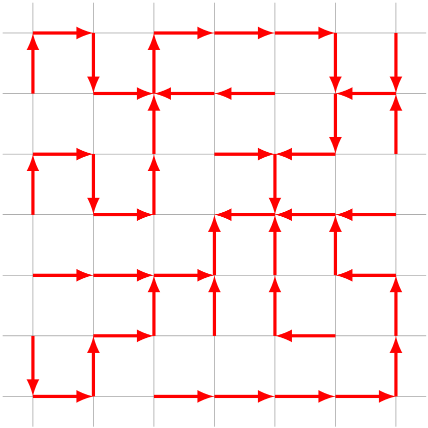
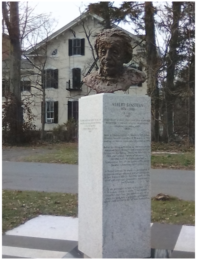
In between random walk and rotor walk

Swee Hong Chan
Cornell University
Joint work with Lila Greco, Lionel Levine, Boyao Li



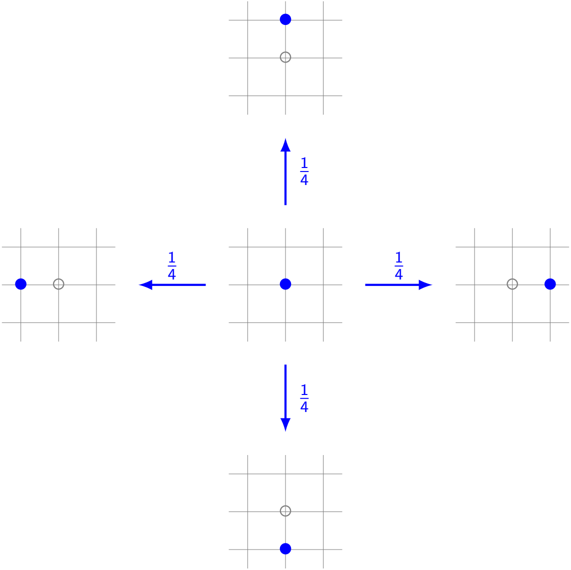

ALBERT EINSTEIN

Random walk

Rotor walk

Simple random walk on $\mathbb{Z}^2$

Simple random walk on $\mathbb{Z}^2$



Simple random walk on $\mathbb{Z}^2$

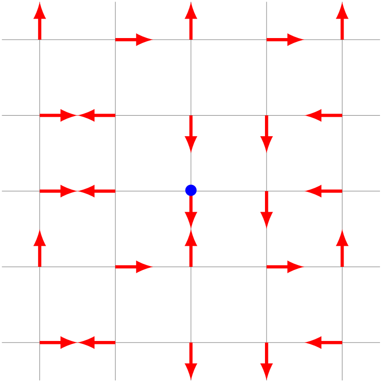
- Visits every site infinitely often? Yes!
- Scaling limit? The standard 2-D Brownian motion:

$$\Big(\frac{1}{\sqrt{n}}\underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}}\Big)_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}\underbrace{(B_1(t), B_2(t))}_{\substack{\text{independent standard}\\ \text{Brownian motions}}}{}_{t\geq 0}.$$

Rotor walk on $\mathbb{Z}^2$

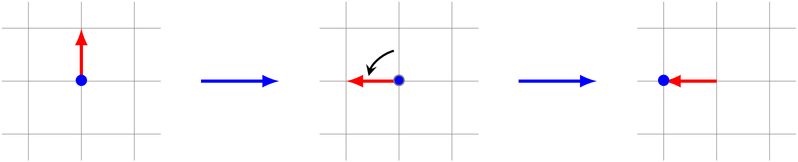
Rotor walk on $\mathbb{Z}^2$

Put a signpost at each site.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

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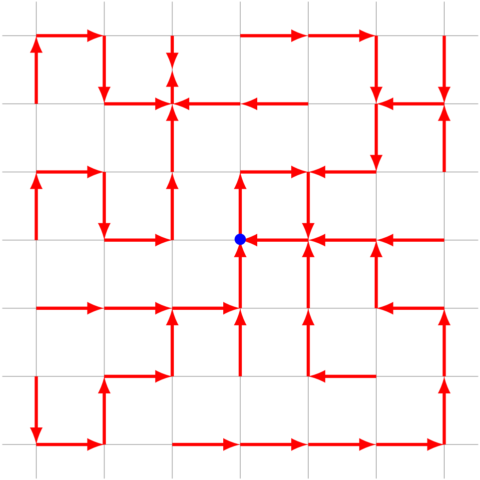

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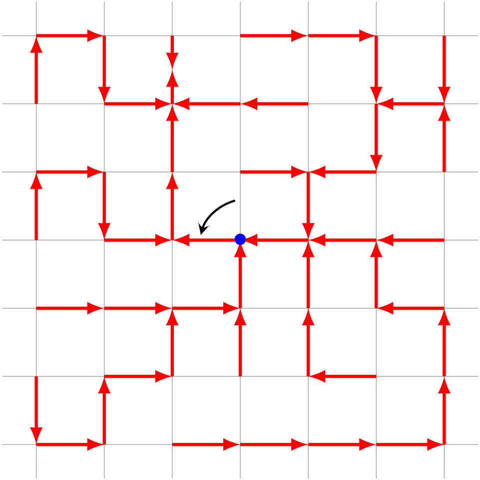

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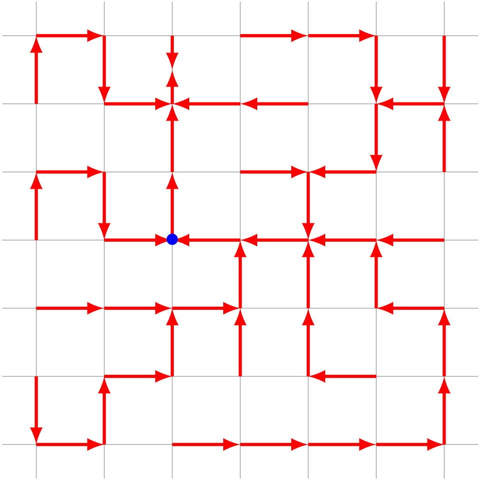

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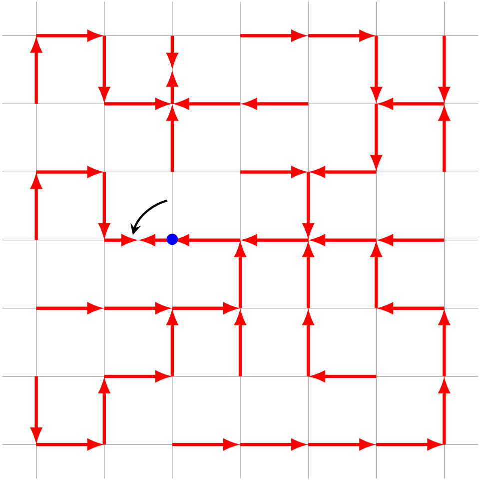

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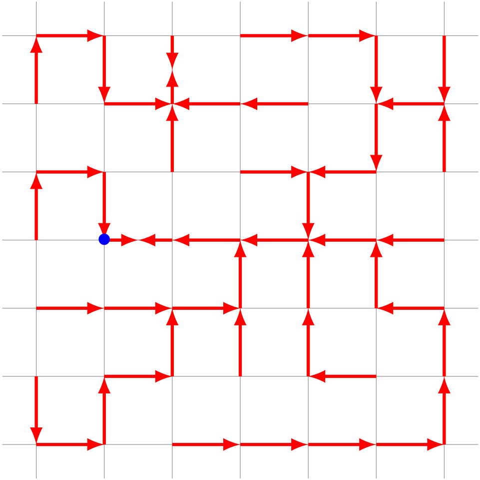

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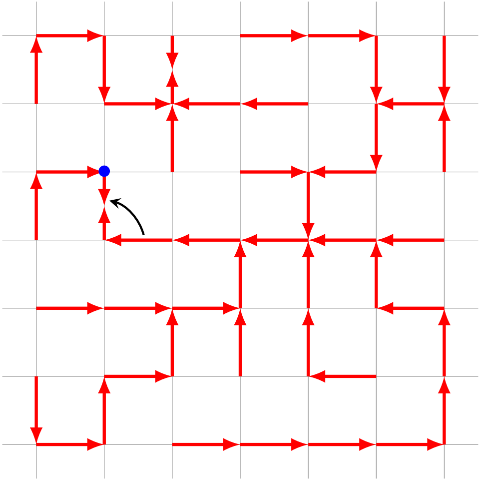

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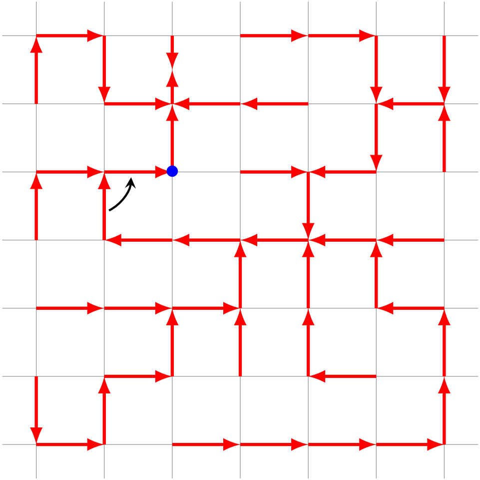

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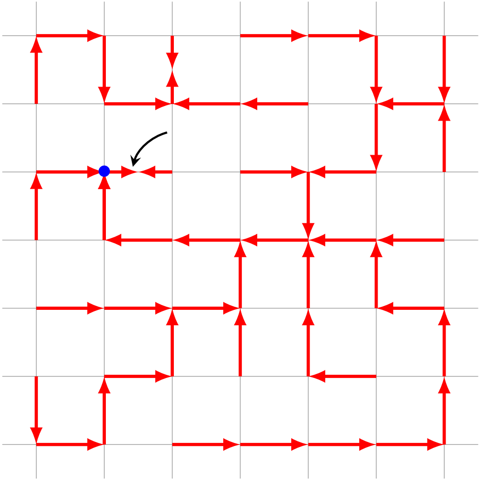

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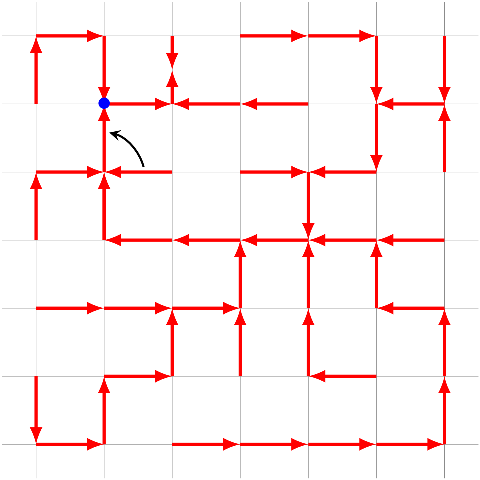

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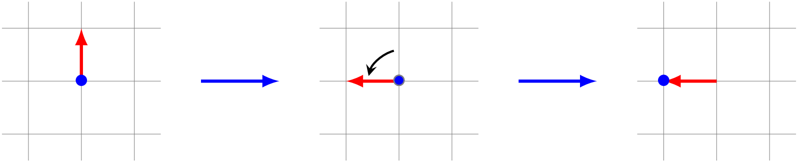

The signpost says:
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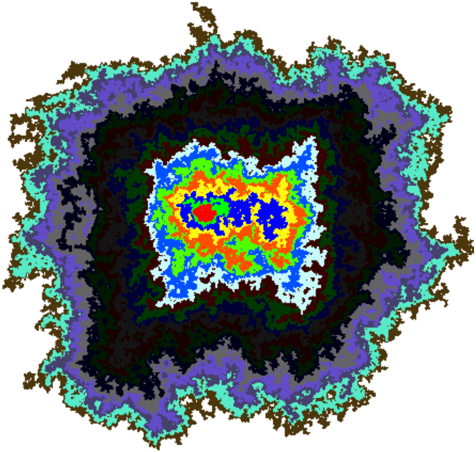
Conjectures for rotor walk on $\mathbb{Z}^2$

If the initial signposts are i.i.d. uniform among the four directions, then

- (PDDK '96) Visits every site infinitely often?
- (PDDK '96) $\#\{X_1, \ldots, X_n\}$ is $\asymp n^{2/3}$?
  (compare with $n/\log n$ for the simple random walk.)
- (Kapri-Dhar '09) The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc?

The set of sites visited by rotor walk on $\mathbb{Z}^2$



$\{X_1, \ldots, X_n\}$ after 80 returns to the origin (by Laura Florescu).

More randomness please!

Well studied

Many open problems

Random

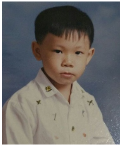
Deterministic

More randomness please!

Well studied

Random

Let's study this!!!

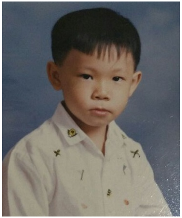


Something in between

Many open problems

Deterministic

p-rotor walk on $\mathbb{Z}^2$



p-rotor walk on $\mathbb{Z}^2$

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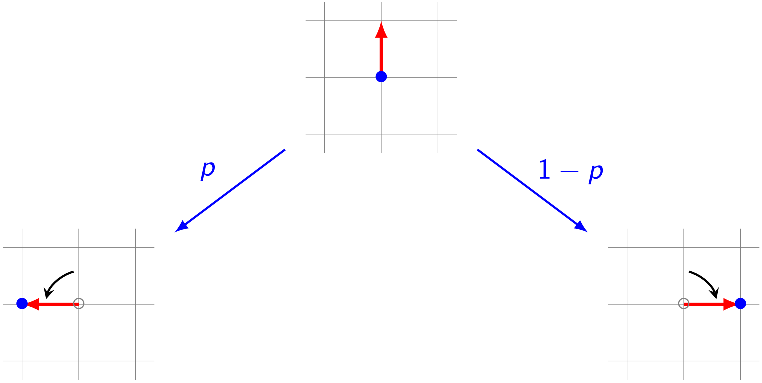
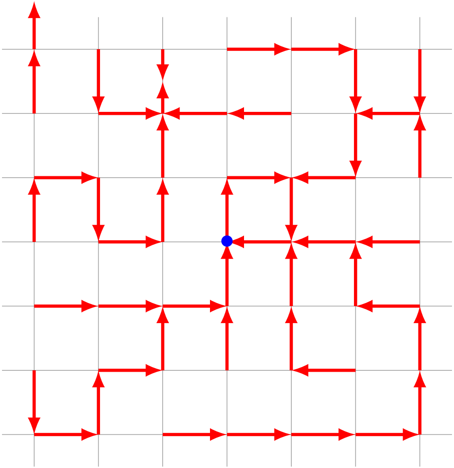
With probability $1 - p$, turn the signpost 90° clockwise.



p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



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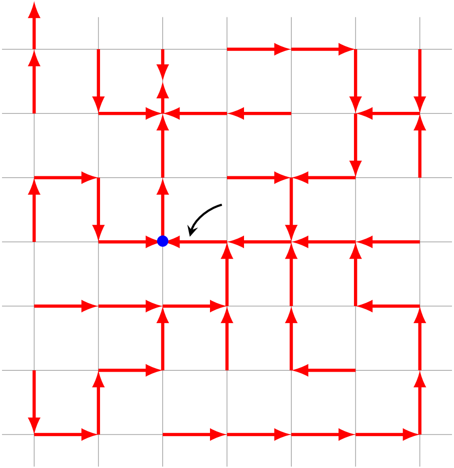

Follow the rule.

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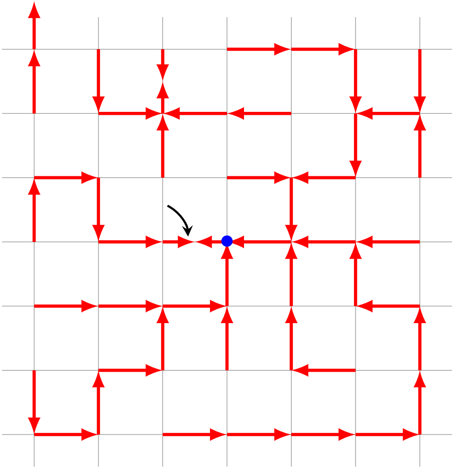

Do the opposite.

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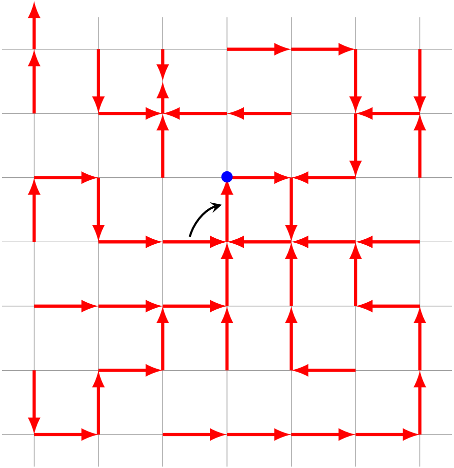
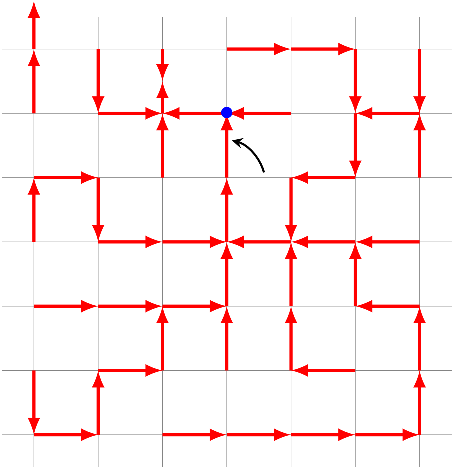


Do the opposite again.

p-rotor walk on $\mathbb{Z}^2$

Follow rotor walk rule with prob. p, do the opposite with prob. $1 - p$



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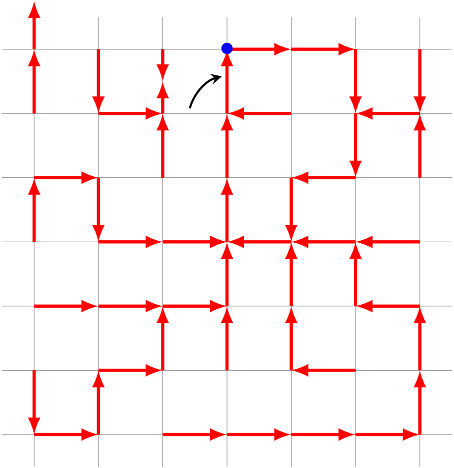

Do the opposite.

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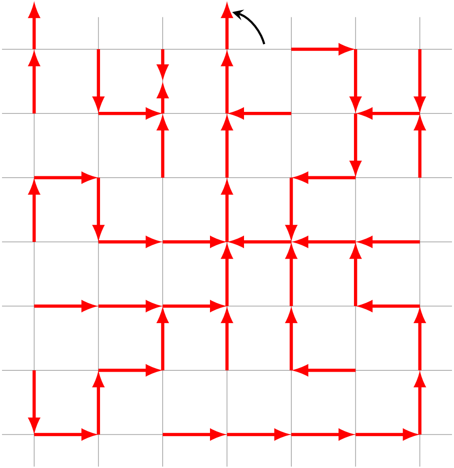

Ops...

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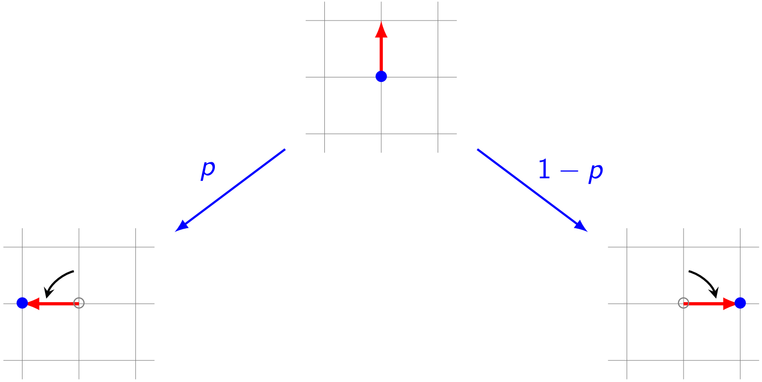
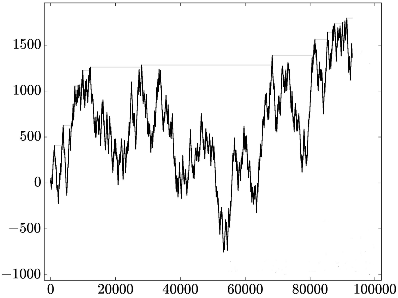


Recover the rotor walk if $p = 1$.

Scaling limit for p-rotor walk on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 18) The scaling limit for p-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$,

$$Y(t) = \underbrace{B(t)}_{\substack{\text{standard}\\ \text{Brownian}\\ \text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{perturbation at}\\ \text{minimum}}}, \qquad t \geq 0.$$



$Y(t)$ for $a = -0.998$, and $b = 0$ (by Wilfried Huss).

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Question: Is the scaling limit for p-rotor walk on $\mathbb{Z}^2$ is a "2-D perturbed Brownian motion"?

Problem: How to define "2-D perturbed Brownian motion"?.

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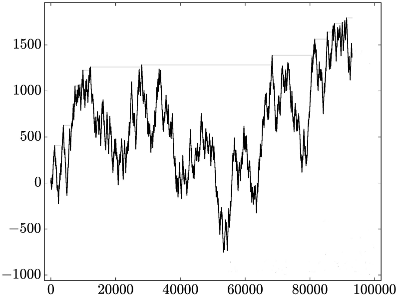

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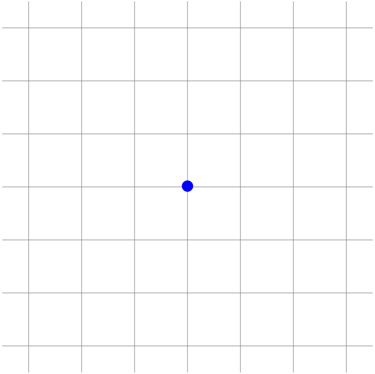
Problem: How to define "2-D perturbed Brownian motion"?.

Conjecture: The scaling limit for p-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Uniform spanning tree plus one edge (UST$^+$)



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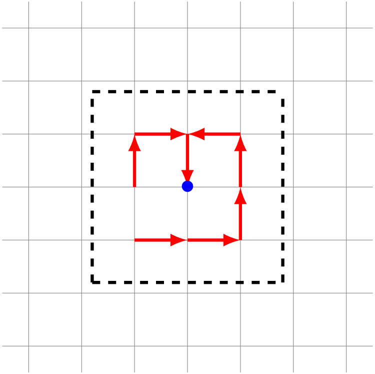
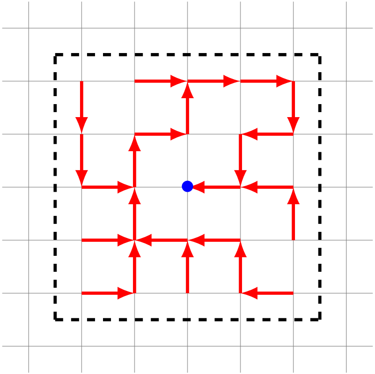


Pick a spanning tree of the black box directed to the origin (uniformly at random).

Uniform spanning tree plus one edge (UST^+)



Take the limit as the black box grows until it covers $\mathbb{Z}^2$.

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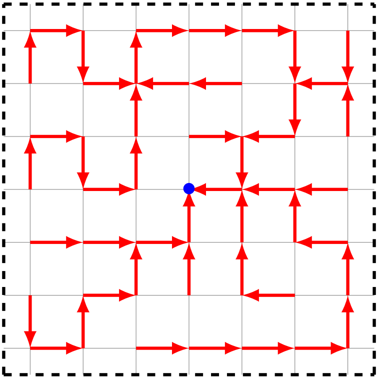

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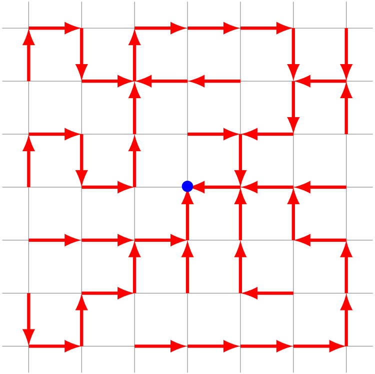

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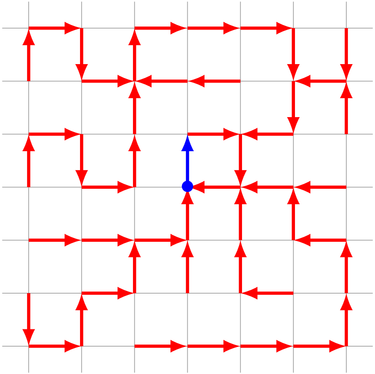

Add a signpost from the origin, uniform among the four directions.

Scaling limit for p-rotor walk on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '18+)

Let $p = \frac{1}{2}$ and let the uniform spanning tree plus one edge be the initial signposts configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the} \\ \text{walker at time } [nt]}})_{t \geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent} \\ \text{Brownian motions}}})_{t \geq 0}.$$

Main ideas of the proof

- How does $p = \frac{1}{2}$ help?

- How does uniform spanning tree plus one edge help?

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  Because then the p-rotor walk is a martingale:

$$\mathbb{E}[\underbrace{X_{t+1}}_{\substack{\text{location of}\\ \text{the walker}}} \mid \mathcal{F}_t] = X_t + (p \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\substack{90^\circ \text{ rotation}\\ \text{matrix}}} + (1-p) \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\substack{-90^\circ \text{ rotation}\\ \text{matrix}}}) \underbrace{\rho_t(X_t)}_{\substack{\text{signpost of } X_t\\ \text{at time } t}}$$

$$= X_t.$$

- How does uniform spanning tree plus one edge help?

Martingale CLT

If $(X_t)_{t\geq 0}$ is a martingale with bounded differences in $\mathbb{R}^2$, then

$$\frac{1}{\sqrt{n}}(X_{[nt]})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent}\\ \text{Brownian motions}}})_{t\geq 0},$$

provided that:

$$\frac{1}{n}\sum_{t=0}^{n-1}(\underbrace{X_{t+1}-X_t}_{\substack{\text{martingale}\\ \text{difference}}})(X_{t+1}-X_t)^\top \xrightarrow[P]{n\to\infty} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \qquad \text{(LLN)}$$

In our case, (LLN) means the fraction of vertical signposts encountered by the walker converges (in probability) to one-half.

Main ideas of the proof

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Main ideas of the proof

- How does $p = \frac{1}{2}$ help?

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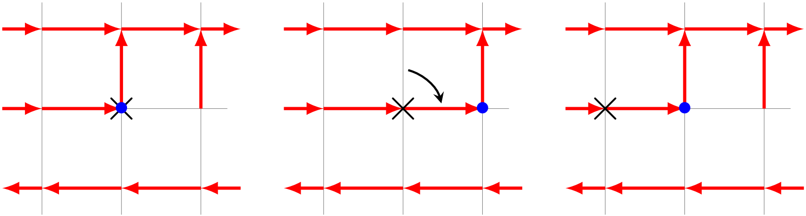
- How does uniform spanning tree plus one edge help?

  Because it is stationary and ergodic from the walker's POV.

Stationarity from the walker's POV

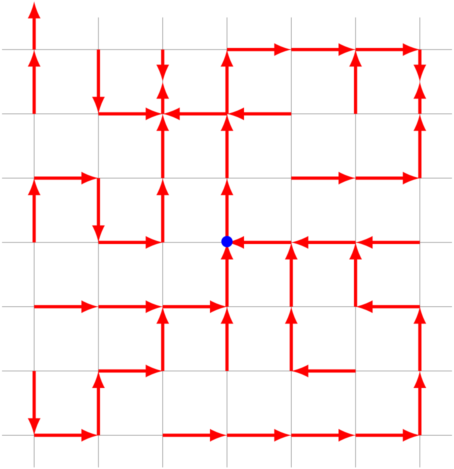
A signposts configuration $(\rho_0(x))_{x\in\mathbb{Z}^2}$ is stationary in time from the walker's point of view if

$$\underbrace{(\widehat{\rho}_1(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signposts conf. at}\\ \text{time 1 from walker's POV}}} := (\rho_1(x-X_1))_{x\in\mathbb{Z}^2} \overset{d}{=} \underbrace{(\rho_0(x))_{x\in\mathbb{Z}^2}}_{\substack{\text{signposts conf.}\\ \text{at time 0}}}.$$



ρ_0 ρ_1 $\widehat{\rho}_1$

Why is UST$^+$ stationary?



The signposts at previously visited sites form a tree oriented towards the walker.

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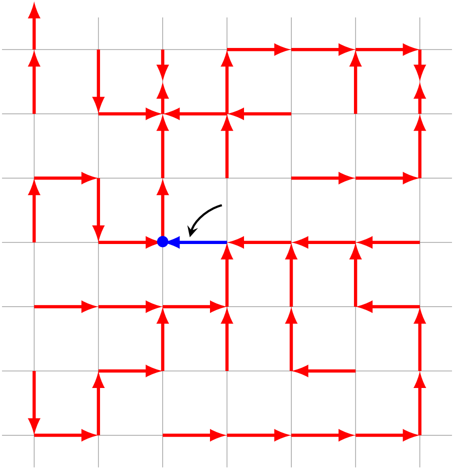

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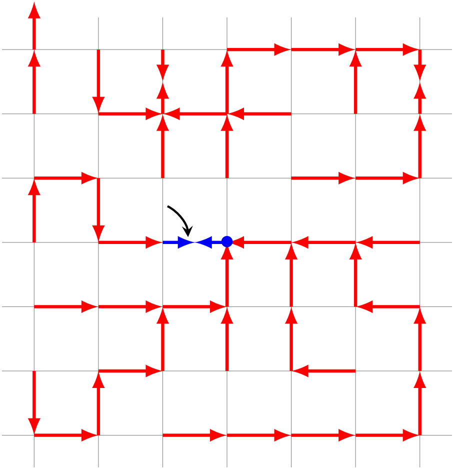

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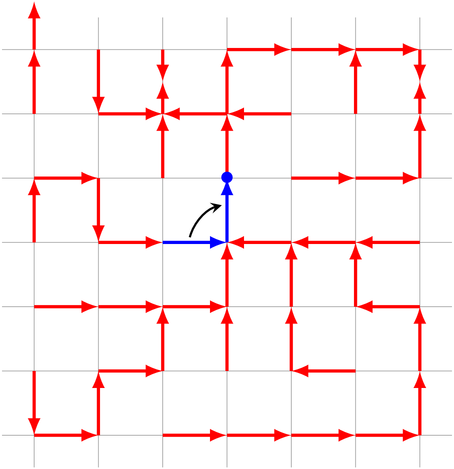

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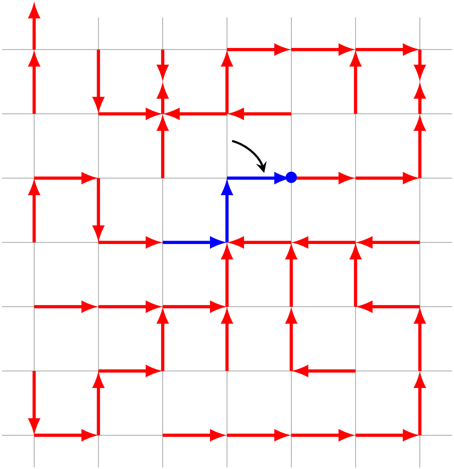

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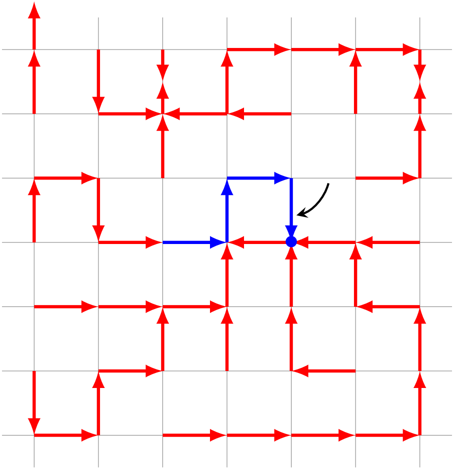

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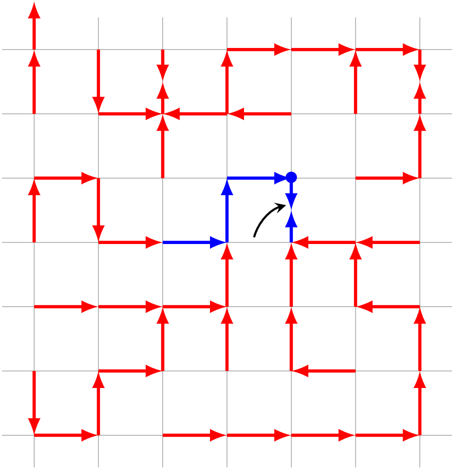

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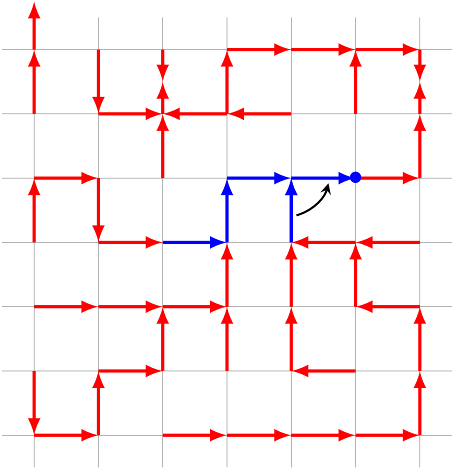

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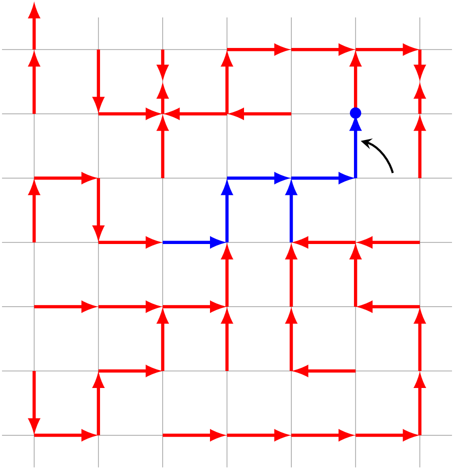

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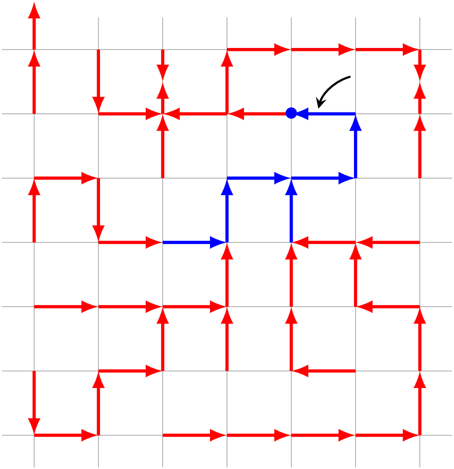

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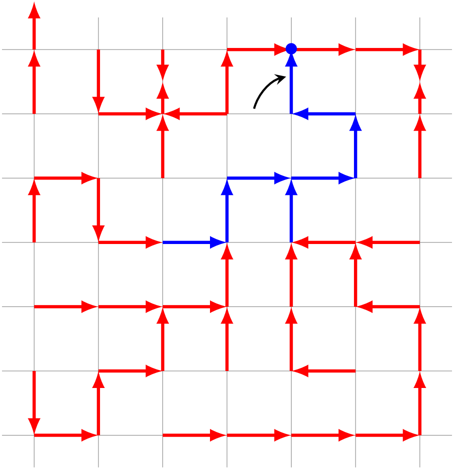
Why is UST$^+$ stationary?



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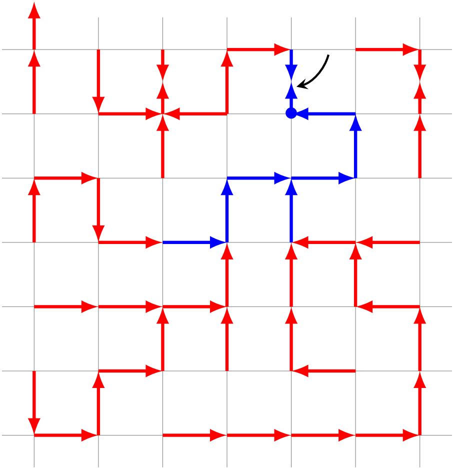

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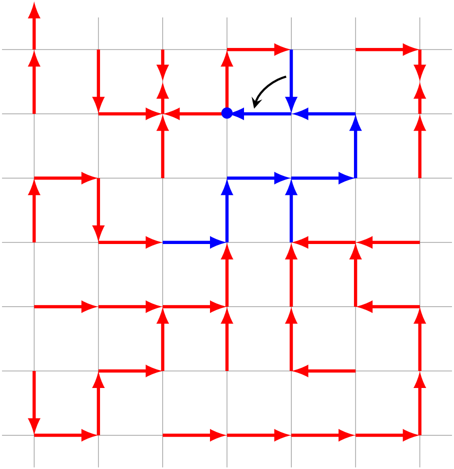

The signposts at previously visited sites form a tree oriented towards the walker.

Ergodicity from the walker's POV

An event A is time-invariant if

$$\{\underbrace{\widehat{\rho} = (\widehat{\rho}_0, \widehat{\rho}_1, \widehat{\rho}_2, \ldots)}_{\substack{\text{signposts configuration} \\ \text{from time 0 to } \infty}} \in A\} = \{\ \widehat{\rho}\ \mid\ \underbrace{(\widehat{\rho}_1, \widehat{\rho}_2, \widehat{\rho}_3, \ldots)}_{\substack{\text{signposts configuration} \\ \text{from time 1 to } \infty}} \in A\}$$

Examples:

- All signposts encountered by the walker are pointing north.
- The walker eventually follows the rule of rotor walk.

A configuration of signposts is ergodic in time if any time-invariant event has probability either 0 or 1.

UST^+ is ergodic in time from the walker's POV

Proposition

The uniform spanning tree plus one edge in $\mathbb{Z}^2$ is ergodic in time.

Key properties of (undirected) uniform spanning trees that we use:

- Undirected UST is ergodic in space w.r.t translations of $\mathbb{Z}^2$.
- Undirected UST of $\mathbb{Z}^2$ is connected a.s..

Back to martingale CLT

If $(X_t)_{t\geq 0}$ is a martingale with bounded differences in $\mathbb{R}^2$, then

$$\frac{1}{\sqrt{n}}(X_{[nt]})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}\underbrace{(B_1(t), B_2(t))}_{\substack{\text{independent}\\ \text{Brownian motions}}}{}_{t\geq 0},$$

provided that:

$$\frac{1}{n}\sum_{t=0}^{n-1}\underbrace{(X_{t+1}-X_t)}_{\substack{\text{martingale}\\ \text{difference}}}(X_{t+1}-X_t)^\top \xrightarrow[P]{n\to\infty} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \qquad \text{(LLN)}$$

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provided that:

$$\frac{1}{n}\sum_{t=0}^{n-1} \mathbf{1}\{\underbrace{\widehat{\rho}_t(0)}_{\substack{\text{walker's signpost}\\ \text{at time } t}} = \text{vertical}\} \xrightarrow[P]{n\to\infty} \frac{1}{2}. \qquad \text{(LLN)}$$

Back to martingale CLT

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Ergodicity of UST^+ implies (LLN) by pointwise ergodic theorem!

Back to the scaling limit

Theorem (C., Greco, Levine, Li '18+)

Let $p = \frac{1}{2}$ and let the uniform spanning tree plus one edge be the initial signposts configuration. Then, with probability 1, the p-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion:

$$\frac{1}{\sqrt{n}}(\underbrace{X_{[nt]}}_{\substack{\text{location of the}\\ \text{walker at time } [nt]}})_{t\geq 0} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}}(\underbrace{B_1(t), B_2(t)}_{\substack{\text{independent}\\ \text{Brownian motions}}})_{t\geq 0}.$$

Key tools that we use:

- Martingale CLT;
- Pointwise ergodic theorem;
- Undirected UST of $\mathbb{Z}^2$ is ergodic in space;
- Undirected UST of $\mathbb{Z}^2$ is connected a.s..

What is next?

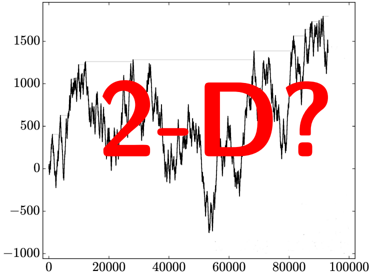
For p-rotor walk with UST^+ as the initial signposts configuration:

What is next?

For p-rotor walk with UST^+ as the initial signposts configuration:

Question: Prove scaling limit for when $p \neq \frac{1}{2}$?

Problem: Need to define the "2-D perturbed Brownian motion (?)".



What is next?

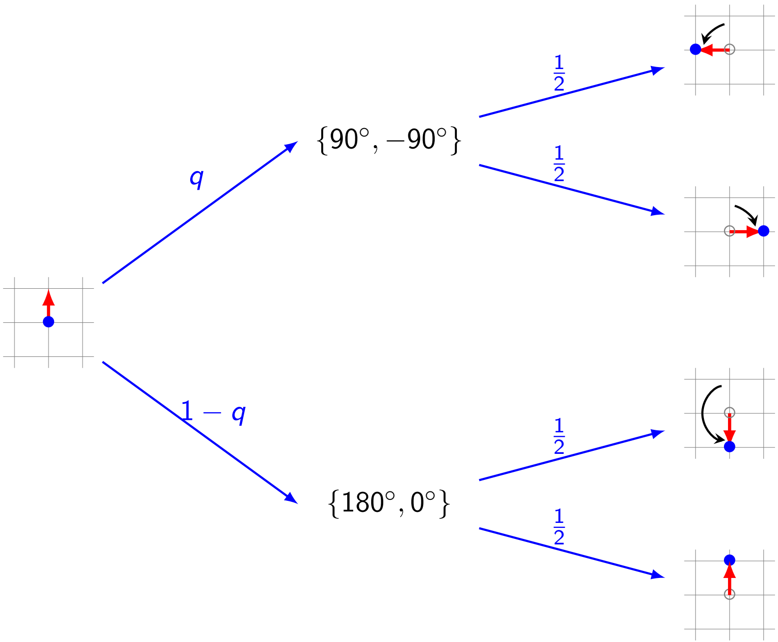
For p-rotor walk with UST^+ as the initial signposts configuration:

Question: Scaling limit for higher dimensions when $p = \frac{1}{2}$?

Answer(?): YES for elliptic walk on $\mathbb{Z}^d$ with $d \in \{2, 3, 4\}$.

Open for elliptic walk on $\mathbb{Z}^d$ when $d \geq 5$.
Problem: Undirected UST of $\mathbb{Z}^d$ for $d \geq 5$ is not connected!

An elliptic walk on $\mathbb{Z}^2$



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What is next?

For p-rotor walk with UST^+ as the initial signposts configuration:

Question: Does the walk visit every site in $\mathbb{Z}^d$ infinitely often?

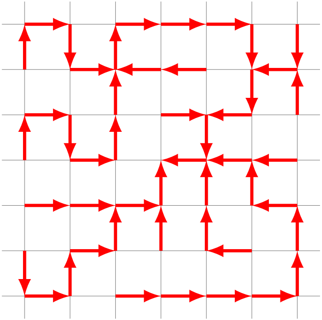
Answer for $p \in \{0, 1\}$: NO (Florescu, Levine, Peres 16):

Answer for $\mathbb{Z}$ with $p \in (0, 1)$: YES (Huss, Levine, Sava-Huss 18).

Answer for $\mathbb{Z}^d$ with $d \geq 5$: NO.
Recurrence implies undirected UST of $\mathbb{Z}^d$ is connected.

Open for $\mathbb{Z}^d$ with $p \in (0, 1)$ and $d \in \{2, 3, 4\}$.

THANK YOU!



Preprint coming soon to an arXiv server near you!

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