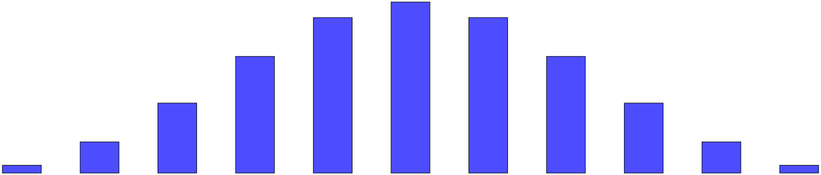
Complexity of Log-concave Inequalities for Matroids

Swee Hong Chan

joint with Igor Pak



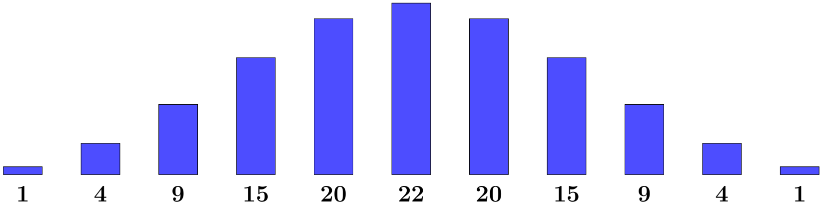
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{N}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad (1 < k < n).$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \;\leq\; \cdots \;\leq\; a_m \;\geq\; \cdots \;\geq\; a_n \quad \text{for some } 1 \leq m \leq n.$$



Log-concave shaped objects in real life

Cheonmachong (천마총) burial mound,
Gyeongju, South Korea.

Example 1: Binomial coefficients

$$a_k \;=\; \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\,a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example 2: Permutation inversion sequence

Let

$$a_k \;:=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; \prod_{i=1}^{n-1} (1 + q + q^2 + \ldots + q^i)$$

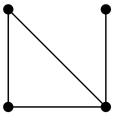
is a product of log-concave polynomials.

Example 3: Mason's conjecture for matroids

Let $\mathcal{M}$ be a matroid, and

a_k := number of independent sets with k elements.

Log-concavity was conjectured for all matroids (Mason '72), and was proved using combinatorial Hodge theory (Adiprasito–Huh–Katz '18).



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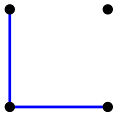
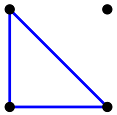
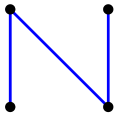


forest



not forest



spanning tree

Example 3: Mason's conjecture for matroids

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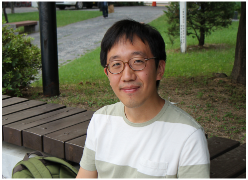
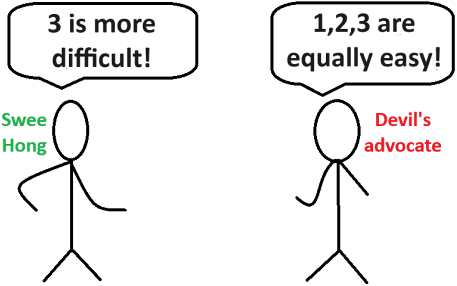


June Huh

Fields Medal

Motivation

Which log-concave inequality is more "difficult"?



Motivation

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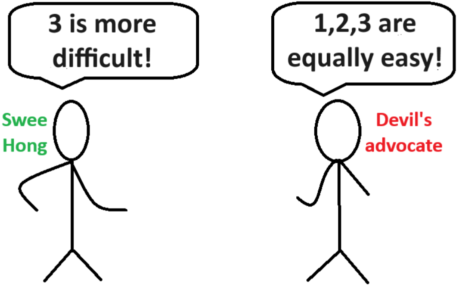


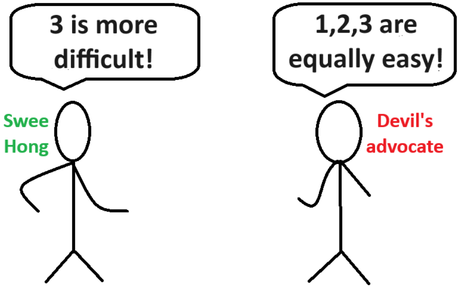
We will show that [redacted] (3)

is **strictly more** difficult than the rest,

using **Complexity Theory**.

Motivation

Which log-concave inequality is more "difficult"?



We will show that a generalization of (3) is **strictly more** difficult than the rest, using **Complexity Theory**.

Stanley–Yan inequality

Stanley–Yan inequality (simple case)

Let $\mathcal{M}$ be a matroid with ground set X and rank r.

Fix a subset S of X. Let

$\mathrm{B}(k) :=$ no. of **bases** B such that $|B \cap S| = k$,

multiplied by $r! \times \binom{r}{k}^{-1}$.

Theorem (Stanley '81, Yan '23)

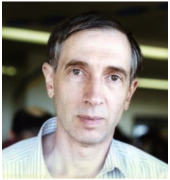
The sequence $\mathrm{B}(1), \mathrm{B}(2), \ldots$ *is log-concave,*

$$\mathrm{B}(k)^2 \geq \mathrm{B}(k+1)\,\mathrm{B}(k-1) \qquad (k \in \mathbb{N}).$$

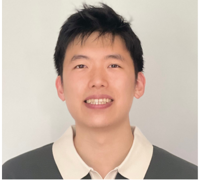
Stanley–Yan inequality (simple)

Theorem (Stanley '81, Yan '23)

$$\mathrm{B}(k)^2 \;\geq\; \mathrm{B}(k+1)\,\mathrm{B}(k-1) \qquad (k \in \mathbb{N}).$$



Richard Stanley



Alan Yan

Stanley–Yan inequality (simple)

Theorem (Stanley '81, Yan '23)

$$\mathrm{B}(k)^2 \;\geq\; \mathrm{B}(k+1)\,\mathrm{B}(k-1) \qquad (k \in \mathbb{N}).$$

Proved for regular matroids by (Stanley '81) using Alexandrov–Fenchel inequality for mixed volumes.

Proved for all matroids by (Yan '23) using theory of Lorentzian polynomials.

Proof of Mason's conjecture using Stanley–Yan inequality

Proof of Mason's conjecture using SY inequality

Let

$\mathcal{M}$:= original matroid in Mason's conjecture;

$\mathcal{F}$:= matroid with r elements and with every subset being independent;

$\mathcal{M}'$:= direct sum of $\mathcal{M}$ and $\mathcal{F}$;

S := ground set of $\mathcal{M}$.

Then

$$\mathrm{I}(k) \text{ for } \mathcal{M} \quad = \quad \frac{1}{r!} \times \mathrm{B}(k) \text{ for } \mathcal{M}'.$$

Proof of Mason's conjecture using SY inequality

Since

$$\mathrm{I}(k) \text{ for } \mathcal{M} \quad = \quad \frac{1}{r!} \times \mathrm{B}(k) \text{ for } \mathcal{M}',$$

we then conclude that

Stanley–Yan inequality for $\mathcal{M}'$
implies Mason's conjecture for $\mathcal{M}$.

Stanley–Yan inequality (full version)

Fix $d \geq 0$, disjoint subsets $S, S_1, \ldots, S_d$ of X, and $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

$$\mathrm{B}_d(k) := \begin{array}{l} \text{number of bases } B \text{ of } \mathcal{M} \text{ such that} \\ |B \cap S| = k, \ |B \cap S_i| = \ell_i \ \text{ for } \ i \in [d], \end{array}$$

multiplied by $r! \times \binom{r}{k, \ell_1, \ldots, \ell_d}^{-1}$.

Theorem (Stanley '81, Yan '23)

The sequence $\mathrm{B}_d(1), \mathrm{B}_d(2), \ldots$ *is log-concave,*

$$\mathrm{B}_d(k)^2 \ \geq \ \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1) \qquad (k \in \mathbb{N}).$$

What we want to do

Theorem (Stanley '81, Yan '23)

The sequence $\mathrm{B}_d(1), \mathrm{B}_d(2), \ldots$ *is log-concave,*

$$\mathrm{B}_d(k)^2 \;\geq\; \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1) \qquad (k \in \mathbb{N}).$$

Both LHS and RHS of this inequality has combinatorial interpretations.

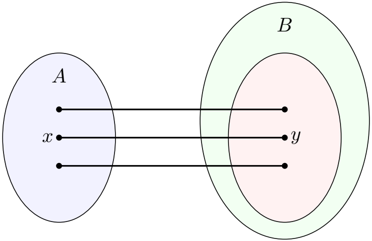
But we will show that this inequality has **no combinatorial injective proof**.

Combinatorial injective proof

Combinatorial injection

An injection $f : A \to B$ is combinatorial if

- Given $x \in A$, the image $f(x)$ is computable in poly($|x|$) steps;
- Given $y \in B$, it takes poly($|y|$) steps to decide if y is in image of f; and if so, the pre-image $f^{-1}(y)$ is computable in poly($|y|$) steps.



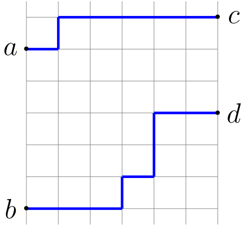
Example: Injective proof of binomial inequality

$$\binom{n}{k}^2 \geq \binom{n}{k+1}\binom{n}{k-1} \qquad (1 < k < n).$$

This inequality has a lattice path interpretation:

$K(a \to c, b \to d) :=$ no. of pairs of north-east lattice paths from a to c and b to d,

for $a, b, c, d \in \mathbb{Z}^2$.



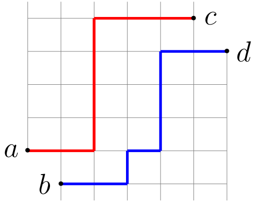
Example: Injective proof of binomial inequality

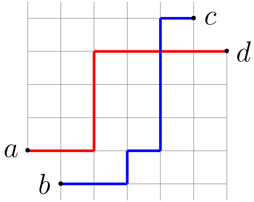
Let

$$a = (0,1), \qquad c = (k, n-k+1),$$
$$b = (1,0), \qquad d = (k+1, n-k).$$

Then

$$K(a \to c, b \to d) = \binom{n}{k}^2,$$
$$K(a \to d, b \to c) = \binom{n}{k-1}\binom{n}{k+1}.$$

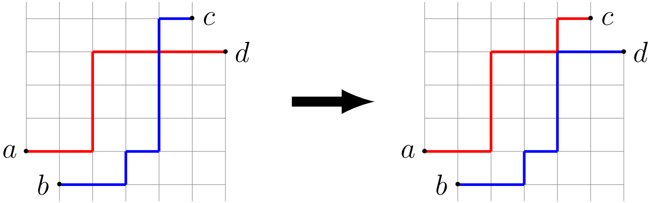




Example: Injective proof of binomial inequality

$$f : K(a \to d, b \to c) \to K(a \to c, b \to d)$$

is defined by path-swapping injections.



Images of f are pairs of lattice paths that intersects.

First main result

Theorem 1 (C.–Pak '24+)
There is **no combinatorial injective proof** *for Stanley–Yan inequality, assuming* $\mathsf{NP}^{\mathsf{NP}} \neq \mathsf{coNP}^{\mathsf{NP}}$.

The assumption above is slightly stronger than $\mathsf{P} \neq \mathsf{NP}$, and is widely used in Complexity Theory.

First main result

Theorem 1 (C.–Pak '24+)
There is **no combinatorial injective proof** *for Stanley–Yan inequality, assuming* $\mathsf{NP}^{\mathsf{NP}} \neq \mathsf{coNP}^{\mathsf{NP}}$.

This result is a consequence of Stanley–Yan inequality being **not in** $\#P$ (explained next slide).

Complexity class $\#P$

Complexity class #P

NP	:=	Problems asking about **existence** of a solution S for input x, where validity of S can be verified in poly($\lvert x \rvert$) time.
#P	:=	Problems asking for **number** of solutions S for input x, where validity of S can be verified in poly($\lvert x \rvert$) time.

Example (Problem in #P)

Count the number of proper 3-colorings of graph G.

Complexity class #P: Equivalent definition

A problem is in #P if, for any input x,

$$\text{Output} \quad = \quad \sum_{S \in \{0,1\}^{\mathrm{poly}(|x|)}} V(x, S)$$

where

$$V(x, S) \quad \in \quad \{0, 1\}$$

can be evaluated in poly$(|x|)$ time.

Note that the size of the output is at most exponential relative to the input x.

Second main result

Consider the following computational problem:

Input: Binary matroid $\mathcal{M}$, subsets $S, S_1, \ldots, S_d$,

integers $k, \ell_1, \ldots, \ell_d$.

Output: $\mathrm{B}_d(k)^2 - \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1)$.

Theorem 2 (C.–Pak '24+)

The problem above **does not belong to** #P, *assuming* $\mathsf{NP}^{\mathsf{NP}} \neq \mathsf{coNP}^{\mathsf{NP}}$.

Second main result

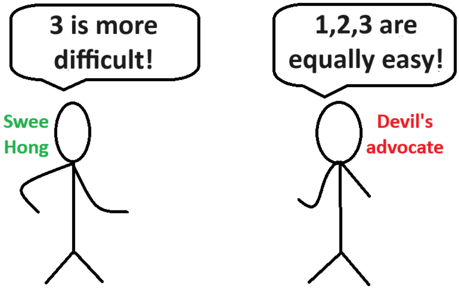
Theorem (C.–Pak '24+)
The problem of computing

$$\mathrm{B}_d(k)^2 - \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1)$$

is **not in** #P*, assuming* $\mathsf{NP}^{\mathsf{NP}} \neq \mathsf{coNP}^{\mathsf{NP}}$*.*

Both LHS and RHS of Stanley–Yan inequality belongs to #P, but their difference does not.

Recall our goal



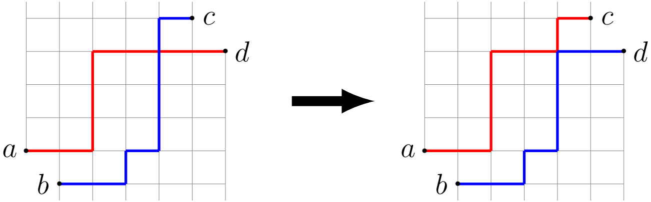
We will now show that Stanley–Yan inequality is strictly more difficult than the binomial inequality and permutation inversion inequality.

Example 1: Binomial inequality

It follows from path-swapping injections that

$$\binom{n}{k}^2 - \binom{n}{k+1}\binom{n}{k-1} \;=\; \text{number of non-intersecting}$$

lattice paths from a to c and b to d.



Thus the defect of this inequality belongs to #P.

Example 2: Permutation inversion inequality

Let a_k = number of $\pi \in S_n$ with k inversions.

$$\text{Then} \quad \sum_{0 \le k \le \binom{n}{2}} a_k \, q^k \; = \; \prod_{i=1}^{n-1} (1 + q + \ldots + q^i)$$

is computable in poly(n) time.

Thus $a_k^2 - a_{k+1} a_{k-1}$ is computable in poly(n) time;

and thus belongs to #P.

Conclusion

We compare three log-concave inequalities:

Binomial inequality: in #P;

Permutation inversion inequality: in #P;

Stanley–Yan inequality: **not in** #P.

This differentiates **Stanley–Yan inequality** from binomial inequality and permutation inversion inequality.

THANK YOU!

Preprint: www.arxiv.org/abs/2407.19608

Webpage: www.math.rutgers.edu/~sc2518/

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Open Problem

Conjecture

Defect of Mason's conjecture

$$\mathrm{I}(k)^2 - \mathrm{I}(k+1)\,\mathrm{I}(k-1) \notin \#\mathrm{P}.$$

We have shown defect of Stanley–Yan inequality does not belong to #P, but not Mason's conjecture.