

# Four Functions Theorem and Schur Positivity

**Swee Hong Chan**

joint with Hong Chen, Igor Pak, and Daniel Soskin

# Four Functions Theorem and Schur Positivity

**Swee Hong Chan**

joint with Hong Chen, Igor Pak, and Daniel Soskin



# Fantasy World Cup 2026

Supercomputer predicts the following probabilities for winning the 2026 Fantasy World Cup:



Argentina  
18%



France  
39%



Spain  
22%



England  
21%

# Fantasy World Cup 2026

Suppose that **Messi suddenly got food poisoning.**

The supercomputer then updates the projected probabilities:



Argentina

9% ↓



France

41% ↑



Spain

24% ↑



England

26% ↑



# Fantasy World Cup 2026



Argentina

9% ↓



France

41% ↑



Spain

24% ↑



England

26% ↑

The event: Messi got food poisoning.

- **Negative** correlation: Argentina's probability of winning World Cup **decreases**.
- **Positive** correlation: Other team's probability of winning World Cup **increases**.

## Correlations: Definition

Events  $A$  and  $B$  are **positively correlated** if

$$P(A \cap B) \geq P(A)P(B).$$

If  $P(B) > 0$ , this is equivalent to

$$P(A \mid B) \geq P(A).$$

That is, knowing  $B$  makes  $A$  **more** likely.

**Negative correlation** is defined by **reversing**  
the inequalities.

## Example: negative correlations in spanning trees

Let  $G$  be a graph.

Let  $e, f$  be distinct edges of  $G$ .

Let  $T$  be a uniform random spanning tree of  $G$ .

Theorem (folklore Kirchhoff 1847)

$\{e \in T\}$  and  $\{f \in T\}$  are *negatively correlated*,

$$P(e, f \in T) \leq P(e \in T)P(f \in T).$$

## Example: negative correlations in spanning trees

Let  $G$  be a graph.

Let  $e, f$  be distinct edges of  $G$ .

Let  $T$  be a uniform random spanning tree of  $G$ .

Theorem (folklore Kirchoff 1847)

$\{e \in T\}$  and  $\{f \in T\}$  are *negatively correlated*,

$$P(e, f \in T) \leq P(e \in T)P(f \in T).$$

**Intuition:** Because  $T$  has a fixed number of edges,  $e$  and  $f$  are competing for the same limited spots.

## Example: negative correlations in spanning trees

Let  $G$  be a graph.

Let  $e, f$  be distinct edges of  $G$ .

Let  $T$  be a uniform random spanning tree of  $G$ .

Theorem (folklore Kirchhoff 1847)

$\{e \in T\}$  and  $\{f \in T\}$  are *negatively correlated*,

$$P(e, f \in T) \leq P(e \in T)P(f \in T).$$

Originally proved via electrical networks;  
modern proofs use determinantal structures.

## Non-example: negative correlations in matroids

Let  $M$  be a matroid of rank  $d$ .

Let  $e, f$  be distinct elements of  $M$ .

Let  $B$  be a uniform random basis of  $M$ .

### Theorem (Seymour–Welsh 1975)

*There exist matroids for which  $\{e \in B\}$  and  $\{f \in B\}$  are **not** negatively correlated.*

Matroids satisfy the finite-slot intuition, and yet do not exhibit negative correlations.

## Non-example: negative correlations in matroids

Let  $M$  be a matroid of rank  $d$ .

Let  $e, f$  be distinct elements of  $M$ .

Let  $B$  be a uniform random basis of  $M$ .

Theorem (Huh, Schröter, Wang, 2022)

$$P(e, f \in B) \leq 2(1 - \frac{1}{d}) P(e \in B)P(f \in B).$$

It remains a major open problem to determine  
if the constant 2 is optimal.

## Example: Erdős–Rényi random graphs

Consider **random graph**  $G(n, p)$  with  $n$  vertices and independent edge probability  $p = \frac{1}{2}$ .

Let  $A = \{G(n, p) \text{ is Hamiltonian}\}$ .

Let  $B = \{G(n, p) \text{ is planar}\}$ .

### Theorem

$A$  and  $B$  are *negatively correlated*.



## Example: Erdős–Rényi random graphs

Consider **random graph**  $G(n, p)$  with  $n$  vertices and independent edge probability  $p = \frac{1}{2}$ .

Let  $A = \{G(n, p) \text{ is Hamiltonian}\}$ .

Let  $B = \{G(n, p) \text{ is planar}\}$ .

### Theorem

$A$  and  $B$  are *negatively correlated*.

**Intuition:** Being Hamiltonian demands **more** edges, but being planar demands **fewer** edges.

## Example: Erdős–Rényi random graphs

Consider **random graph**  $G(n, p)$  with  $n$  vertices and independent edge probability  $p = \frac{1}{2}$ .

Let  $A = \{G(n, p) \text{ is Hamiltonian}\}$ .

Let  $B = \{G(n, p) \text{ is planar}\}$ .

### Theorem

$A$  and  $B$  are *negatively correlated*.

This result can be proved rigorously using the **Ahlsvede–Daykin inequality**.

**Ahlsvede–Daykin inequality  
(Four Functions Theorem)**

# Ahlsvede–Daykin (AD) inequality



## Theorem (Ahlsvede–Daykin 1978)

Let  $a, b, c, d : \mathbb{Z}^\ell \rightarrow \mathbb{R}_{\geq 0}$  be nonnegative functions satisfying

$$a(x)b(y) \leq c(x \vee y)d(x \wedge y) \quad \forall x, y \in \mathbb{Z}^\ell.$$

Then

$$\left( \sum_{x \in \mathbb{Z}^\ell} a(x) \right) \left( \sum_{x \in \mathbb{Z}^\ell} b(x) \right) \leq \left( \sum_{x \in \mathbb{Z}^\ell} c(x) \right) \left( \sum_{x \in \mathbb{Z}^\ell} d(x) \right).$$

Here  $\vee$  is pointwise maximum and  
 $\wedge$  is pointwise minimum.

# Ahlswe–Daykin (AD) inequality



## Theorem (Ahlswe–Daykin 1978)

Let  $a, b, c, d : \mathbb{Z}^\ell \rightarrow \mathbb{R}_{\geq 0}$  be nonnegative functions satisfying

$$a(x)b(y) \leq c(x \vee y)d(x \wedge y) \quad \forall x, y \in \mathbb{Z}^\ell.$$

Then

$$\left( \sum_{x \in \mathbb{Z}^\ell} a(x) \right) \left( \sum_{x \in \mathbb{Z}^\ell} b(x) \right) \leq \left( \sum_{x \in \mathbb{Z}^\ell} c(x) \right) \left( \sum_{x \in \mathbb{Z}^\ell} d(x) \right).$$

Intuition: **Local** log-supermodularity implies  
**global** log-supermodularity.

## Anecdotes: AD inequality

### Quote (Ahlsvede)

*"When Daykin visited Bielefeld, I was on a ladder wallpapering as Daykin explained a complicated new inequality. I instead proposed a simpler, more general version from the top of the ladder: the birth of the Ahlsvede–Daykin inequality."*



Ahlsvede



Daykin

## Anecdotes: AD inequality

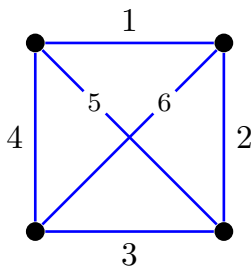
### Quote (Ahlsvede)

*"When Daykin visited Bielefeld, I was on a ladder wallpapering as Daykin explained a complicated new inequality. I instead proposed a simpler, more general version from the top of the ladder: the birth of the Ahlsvede–Daykin inequality."*

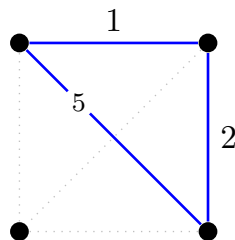


## Proof: Hamiltonian-planar correlation

Each  $x \in \{0, 1\}^{\binom{n}{2}}$  corresponds to a possible outcome of  $G(n, p)$ .



Base graph  $K_4$



$x = (1, 1, 0, 0, 1, 0)$



## Proof: Planar-Hamiltonian correlation

Let  $a, b, c, d : \{0, 1\}^{\binom{n}{2}} \rightarrow \mathbb{R}$  be given by

$$a(x) := \mathbf{1}\{x \text{ is Hamiltonian and planar}\};$$

$$b(x) := 1$$

$$c(x) := \mathbf{1}\{x \text{ is Hamiltonian}\};$$

$$d(x) := \mathbf{1}\{x \text{ is planar}\}.$$

## Proof: Hamiltonian-planar correlation

Being **Hamiltonian** is a **increasing** property:

$x$  is Hamiltonian  $\implies x \vee y$  is Hamiltonian.

Being **planar** is an **decreasing** property:

$x$  is planar  $\implies x \wedge y$  is planar.

This implies, for all  $x, y \in \{0, 1\}^{\binom{n}{2}}$ ,

$$\underbrace{a(x)}_{\substack{\text{Hamiltonian,} \\ \text{planar}}} b(y) \leq \underbrace{c(x \vee y)}_{\text{Hamiltonian}} \underbrace{d(x \wedge y)}_{\text{planar}}.$$

## Proof: Hamiltonian-planar correlation

$$\underbrace{a(x)}_{\substack{\text{Hamiltonian,} \\ \text{planar}}} b(y) \leq \underbrace{c(x \vee y)}_{\text{Hamiltonian}} \underbrace{d(x \wedge y)}_{\text{planar}}$$

is because:

- if  $a(x)b(y) = 0$ , then LHS is 0, and inequality follows.
- If  $a(x)b(y) > 0$ :
  - Then  $x$  is Hamiltonian and planar;
  - Then  $x \vee y$  is Hamiltonian, and  $x \wedge y$  is planar;
  - Then  $a(x) = b(y) = c(x \vee y) = d(x \wedge y) = 1$ , so inequality follows.

## Proof: Hamiltonian-planar correlation

AD inequality thus gives us:

$$\underbrace{\left(\sum_x a(x)\right)}_{2^{\binom{n}{2}} P(A \cap B)} \underbrace{\left(\sum_x b(x)\right)}_{2^{\binom{n}{2}}} \leq \underbrace{\left(\sum_x c(x)\right)}_{2^{\binom{n}{2}} P(A)} \underbrace{\left(\sum_x d(x)\right)}_{2^{\binom{n}{2}} P(B)},$$

where

$A := \{G(n.p) \text{ is Hamiltonian}\}.$

$B := \{G(n.p) \text{ is planar}\}.$

Therefore  $A$  and  $B$  are **negatively correlated**.

*Q.E.D.*

AD inequality is a remarkably simple tool,  
yet yields powerful results across  
probability, order theory, extremal combinatorics.

Today's focus: its applications to  
**Schur polynomials.**

# Schur polynomials

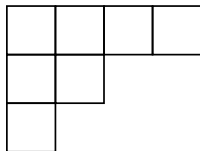
# Integer partitions



Integer partition is a vector  $\lambda \in \mathbb{Z}^\infty$  with finitely many nonzero entries, satisfying

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0.$$

Each partition visually corresponds to a Young diagram.



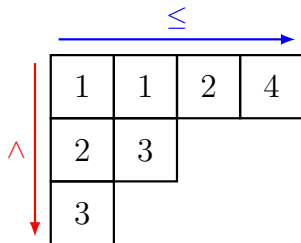
$$\lambda = (4, 2, 1, 0, \dots)$$

We denote by  $\mathbb{Y}$  the set of all integer partitions.

## Semi-standard Young tableau (SSYT)

An **SSYT** of shape  $\lambda$  and weight  $\mu$  is a filling of  $\lambda$  where

- Rows are weakly increasing;
- Columns are strictly increasing;
- Each integer  $i$  appears exactly  $\mu_i$  times.



$$\lambda = (4, 2, 1), \mu = (2, 2, 2, 1).$$



# Schur polynomials

For a partition  $\lambda$ , Schur polynomial  $s_\lambda := s_\lambda(\mathbf{x})$  is

$$s_\lambda := \sum_{T \in \text{SSYT}(\lambda)} x^{\mu(T)},$$

where  $\mu(T)$  is the weight of the tableau  $T$ .

# Schur polynomials

For a partition  $\lambda$ , **Schur polynomial**  $s_\lambda := s_\lambda(\mathbf{x})$  is

$$s_\lambda := \sum_{T \in \text{SSYT}(\lambda)} x^{\mu(T)},$$

where  $\mu(T)$  is the weight of the tableau  $T$ .

- They are **characters** of irreducible representations of the general linear groups.
- They are **symmetric**, and form a **linear basis** for the space of all symmetric polynomials.

Schur polynomials serve as a **natural bridge** connecting algebra, combinatorics, and representation theory.

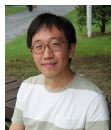
Today, we focus on the **log-concavity** phenomena exhibited by Schur polynomials.

**Log-concavity and Schur polynomials:  
First phenomenon**

# First log-concavity phenomenon (informal)

Theorem (Huh–Matherne–Mészáros–St.  
Dizier 2022)

*Coefficients of Schur polynomial  $s_\lambda$  are log-concave.*



Huh



Matherne



Mészáros



St. Dizier

## HMMS inequality

For partition  $\lambda$  and vector  $\mu$ , the Kostka number  $K_{\lambda,\mu}$  is

$$s_{\lambda} =: \sum_{\mu} K_{\lambda,\mu} x^{\mu}.$$

the coefficient of  $x^{\mu}$  in monomial expansion of  $s_{\lambda}$ .

### Theorem (HMMS inequality)

*For any integers  $i$  and  $j$ ,*

$$K_{\lambda,\mu+e_i-e_j} K_{\lambda,\mu+e_j-e_i} \leq K_{\lambda,\mu}^2,$$

*where  $e_i$  is the  $i$ -th standard unit vector.*

## HMMS inequality (continued)

The HMMS inequality was proved by bringing to light the Lorentzian polynomial structure hidden inside Schur polynomials.

## HMMS inequality (continued)

The HMMS inequality was proved by bringing to light the Lorentzian polynomial structure hidden inside Schur polynomials.

问渠那得清如许？为有源头活水来。

“How can the pool achieve such perfect clarity?  
Because a fresh spring flows from deep beneath it.”

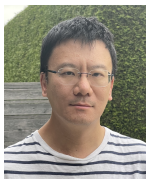


**Log-concavity and Schur polynomials:  
Second phenomenon**

## Second log-concavity phenomenon (informal)

Theorem (Lam–Postnikov–Pylyavskyy 2007)

*Schur polynomial  $s_\lambda$  act as a log-concave function with respect to the partition  $\lambda$ .*



Lam



Postnikov



Pylyavskyy

# Okounkov inequality



For symmetric polynomials  $f, g$ , we write

$f \leq_s g \iff$  Schur expansion of  $g - f$   
has nonnegative coefficients.

Theorem (Lam–Postnikov–Pylyavskyy 2007)

For any partitions  $\lambda$  and  $\mu$ ,

$$s_\lambda s_\mu \leq_s s_{\lceil \frac{\lambda+\mu}{2} \rceil} s_{\lfloor \frac{\lambda+\mu}{2} \rfloor},$$

where  $\lceil \cdot \rceil$  and  $\lfloor \cdot \rfloor$  are defined coordinate-wise.

# Okounkov inequality

Theorem (Lam–Postnikov–Pylyavskyy 2007)

For any partitions  $\lambda$  and  $\mu$ ,

$$s_\lambda s_\mu \leq_s s_{\lceil \frac{\lambda+\mu}{2} \rceil} s_{\lfloor \frac{\lambda+\mu}{2} \rfloor},$$

where  $\lceil \cdot \rceil$  and  $\lfloor \cdot \rfloor$  are defined coordinate-wise.

Okounkov (1997) established ordinary positivity, leaving the more challenging Schur positivity as a conjecture.

# Okounkov inequality

## Theorem (Lam–Postnikov–Pylyavskyy 2007)

For any partitions  $\lambda$  and  $\mu$ ,

$$s_\lambda s_\mu \leq_s s_{\lceil \frac{\lambda+\mu}{2} \rceil} s_{\lfloor \frac{\lambda+\mu}{2} \rfloor},$$

where  $\lceil \cdot \rceil$  and  $\lfloor \cdot \rfloor$  are defined coordinate-wise.

This result was proved by establishing a **Schur log-supermodularity** property, now known as the **LPP inequality**.

# LPP inequality



Theorem (Lam–Postnikov–Pylyavskyy 2007)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$s_{\lambda} s_{\mu} \leq_s s_{\lambda \vee \mu} s_{\lambda \wedge \mu},$$

*where  $\vee$  and  $\wedge$  are defined coordinate-wise.*

LPP inequality was established using  
Temperley–Lieb immanant theory.

# Log-supermodularity implies log-concavity



## Lemma

Let  $f$  be a *shift-invariant* function mapping partitions to symmetric polynomials. If

$$f(\lambda)f(\mu) \leq_s f(\lambda \vee \mu)f(\lambda \wedge \mu) \quad \text{for all } \lambda, \mu \in \mathbb{Y},$$

then

$$f(\lambda) f(\mu) \leq_s f\left(\left\lceil \frac{\lambda + \mu}{2} \right\rceil\right) f\left(\left\lfloor \frac{\lambda + \mu}{2} \right\rfloor\right) \quad \text{for all } \lambda, \mu \in \mathbb{Y}.$$

# Log-supermodularity implies log-concavity



## Lemma

Let  $f$  be a *shift-invariant* function mapping partitions to symmetric polynomials. If

$$f(\lambda)f(\mu) \leq_s f(\lambda \vee \mu)f(\lambda \wedge \mu) \quad \text{for all } \lambda, \mu \in \mathbb{Y},$$

then

$$f(\lambda) f(\mu) \leq_s f\left(\left\lceil \frac{\lambda + \mu}{2} \right\rceil\right) f\left(\left\lfloor \frac{\lambda + \mu}{2} \right\rfloor\right) \quad \text{for all } \lambda, \mu \in \mathbb{Y}.$$

This lemma + LPP inequality = Okounkov inequality.

*Q.E.D.*



The development of Lorentzian polynomial theory to prove log-concavity led to a streamlined proof of HMMS inequality.

The development of Lorentzian polynomial theory to prove log-concavity led to a streamlined proof of HMMS inequality.

Is there a structural framework that leads to a streamlined proof of LPP inequality?

The development of Lorentzian polynomial theory to prove log-concavity led to a streamlined proof of HMMS inequality.

Is there a structural framework that leads to a streamlined proof of LPP inequality?

We introduce Ahlswede–Daykin–Schur inequality as a tool to prove Schur positivity, yielding a streamlined proof of LPP inequality.

## **Ahlsvede–Daykin–Schur inequality**

# Ahlsvede–Daykin–Schur inequality



## Theorem (C.–Chen–Pak–Soskin 2026+)

Let  $a, b, c, d : \mathbb{Y} \rightarrow \mathbb{R}_{\geq 0}$  be nonnegative functions satisfying

$$a(\alpha)b(\beta) \leq c(\alpha \vee \beta)d(\alpha \wedge \beta) \quad \forall \alpha, \beta \in \mathbb{Y}.$$

Then

$$\left( \sum_{\alpha \in \mathbb{Y}} a(\alpha) \right) \left( \sum_{\alpha \in \mathbb{Y}} b(\alpha) \right) \leq \left( \sum_{\alpha \in \mathbb{Y}} c(\alpha) \right) \left( \sum_{\alpha \in \mathbb{Y}} d(\alpha) \right).$$

# Ahlsvede–Daykin–Schur inequality



## Theorem (C.–Chen–Pak–Soskin 2026+)

Let  $a, b, c, d : \mathbb{Y} \rightarrow \mathbb{R}_{\geq 0}$  be nonnegative functions satisfying

$$a(\alpha)b(\beta) \leq c(\alpha \vee \beta)d(\alpha \wedge \beta) \quad \forall \alpha, \beta \in \mathbb{Y}.$$

Then

$$\left( \sum_{\alpha \in \mathbb{Y}} a(\alpha) s_{\alpha} \right) \left( \sum_{\alpha \in \mathbb{Y}} b(\alpha) s_{\alpha} \right) \leq_s \left( \sum_{\alpha \in \mathbb{Y}} c(\alpha) s_{\alpha} \right) \left( \sum_{\alpha \in \mathbb{Y}} d(\alpha) s_{\alpha} \right).$$

Identical hypothesis as **classical AD inequality**, but  
the conclusion now yields **Schur positivity**.

# Ahlsvede–Daykin–Schur inequality



## Theorem (C.–Chen–Pak–Soskin 2026+)

Let  $a, b, c, d : \mathbb{Y} \rightarrow \mathbb{R}_{\geq 0}$  be nonnegative functions satisfying

$$a(\alpha)b(\beta) \leq c(\alpha \vee \beta)d(\alpha \wedge \beta) \quad \forall \alpha, \beta \in \mathbb{Y}.$$

Then

$$\left( \sum_{\alpha \in \mathbb{Y}} a(\alpha) s_{\alpha} \right) \left( \sum_{\alpha \in \mathbb{Y}} b(\alpha) s_{\alpha} \right) \leq_s \left( \sum_{\alpha \in \mathbb{Y}} c(\alpha) s_{\alpha} \right) \left( \sum_{\alpha \in \mathbb{Y}} d(\alpha) s_{\alpha} \right).$$

We now apply this inequality to obtain a streamlined proof of **LPP inequality**.

## Proof of LPP inequality

Let  $\lambda$  and  $\mu$  be partitions in **LPP inequality**. Set

$$\begin{aligned}a(\alpha) &:= \mathbf{1}\{\alpha = \lambda\}; & b(\alpha) &:= \mathbf{1}\{\alpha = \mu\}; \\c(\alpha) &:= \mathbf{1}\{\alpha = \lambda \vee \mu\}; & d(\alpha) &:= \mathbf{1}\{\alpha = \lambda \wedge \mu\}.\end{aligned}$$

**ADS inequality** then implies

$$\underbrace{\left(\sum_{\alpha \in \mathbb{Y}} a(\alpha) s_{\alpha}\right)}_{s_{\lambda}} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} b(\alpha) s_{\alpha}\right)}_{s_{\mu}} \leq_s \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} c(\alpha) s_{\alpha}\right)}_{s_{\lambda \vee \mu}} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} d(\alpha) s_{\alpha}\right)}_{s_{\lambda \wedge \mu}}.$$

*Q.E.D.*



**ADS inequality application:  
Mihalcea conjecture**

# (Stable symmetric) Grothendieck polynomials

For partition  $\lambda$ , the Grothendieck polynomial is

$$G_{\lambda}(\mathbf{x}) := \sum_{\mu \in R(\lambda)} (-1)^{|\mu| - |\lambda|} g_{\mu/\lambda} s_{\mu}(\mathbf{x}),$$

where

- $R(\lambda)$  contains partitions  $\mu$  satisfying

$$\lambda_i \leq \mu_i \leq \lambda_i + i - 1 \quad \text{for all } i.$$

- $g_{\mu/\lambda}$  is the number of tableaux of  $\mu/\lambda$  that are
  - Strictly increasing in rows and columns;
  - Every entry in  $i$ -th row is restricted to  $\{1, \dots, i-1\}$ .

# Properties of Grothendieck polynomials

- They represent **K-theory classes** of Schubert varieties.
- They are **symmetric**, and form a **linear basis** for the space of all symmetric polynomials.

# Properties of Grothendieck polynomials

- They represent **K-theory classes** of Schubert varieties.
- They are **symmetric**, and form a **linear basis** for the space of all symmetric polynomials.

It is thus natural to expect  
Grothendieck polynomials to satisfy  
**log-supermodularity** and **log-concavity**,  
just like Schur polynomials.

## Mihalcea conjecture

Let  $\tilde{G}_\lambda(\mathbf{x}) := (-1)^{|\lambda|} G_\lambda(-\mathbf{x})$ .

Theorem (C.–Chen–Pak–Soskin 2026+)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_s \tilde{G}_{\lambda \vee \mu}(\mathbf{x}) \tilde{G}_{\lambda \wedge \mu}(\mathbf{x}).$$

This establishes **Schur log-supermodularity** for  
Grothendieck polynomials.

## Mihalcea conjecture

Let  $\tilde{G}_\lambda(\mathbf{x}) := (-1)^{|\lambda|} G_\lambda(-\mathbf{x})$ .

Theorem (C.–Chen–Pak–Soskin 2026+)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_s \tilde{G}_{\lambda \vee \mu}(\mathbf{x}) \tilde{G}_{\lambda \wedge \mu}(\mathbf{x}).$$

As a corollary, we obtain **Schur log-concavity** for  
Grothendieck polynomials:

$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_s \tilde{G}_{\lceil \frac{\lambda+\mu}{2} \rceil}(\mathbf{x}) \tilde{G}_{\lfloor \frac{\lambda+\mu}{2} \rfloor}(\mathbf{x}).$$

## Mihalcea conjecture

Let  $\tilde{G}_\lambda(\mathbf{x}) := (-1)^{|\lambda|} G_\lambda(-\mathbf{x})$ .

Theorem (C.–Chen–Pak–Soskin 2026+)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_s \tilde{G}_{\lambda \vee \mu}(\mathbf{x}) \tilde{G}_{\lambda \wedge \mu}(\mathbf{x}).$$

We apply **ADS inequality** to give a streamlined proof of this conjecture.

## Proof of Mihalcea conjecture

$$a(\alpha) := \mathbf{1}\{\alpha \in R(\lambda)\} g_{\alpha/\lambda};$$

$$b(\alpha) := \mathbf{1}\{\alpha \in R(\mu)\} g_{\alpha/\mu};$$

$$c(\alpha) := \mathbf{1}\{\alpha \in R(\lambda \vee \mu)\} g_{\alpha/(\lambda \vee \mu)};$$

$$d(\alpha) := \mathbf{1}\{\alpha \in R(\lambda \wedge \mu)\} g_{\alpha/(\lambda \wedge \mu)}.$$

ADS inequality then implies

$$\underbrace{\left(\sum_{\alpha \in \mathbb{Y}} a(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\lambda}(\mathbf{x})} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} b(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\mu}(\mathbf{x})} \leq_s \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} c(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\lambda \vee \mu}(\mathbf{x})} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} d(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\lambda \wedge \mu}(\mathbf{x})}.$$

*Q.E.D.*



# Applications of ADS inequality

We obtain Schur log-supermodularity and Schur log-concavity for

- Schur polynomials (LPP inequality);
- Grothendieck polynomials (Mihalcea conjecture);
- Dual Grothendieck polynomials;
- Shadow Schur polynomials.

# Applications of ADS inequality

We obtain Schur log-supermodularity and Schur log-concavity for

- Schur polynomials (LPP inequality);
- Grothendieck polynomials (Mihalcea conjecture);
- Dual Grothendieck polynomials;
- Shadow Schur polynomials.

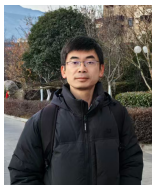
Recently, researchers from Nankai University drew upon ADS inequality to prove new results.

# Guo–Kang–Liu theorem

## Theorem (Guo–Kang–Liu 2026+)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$H_{\lambda}(\mathbf{x}, t, w) H_{\mu}(\mathbf{x}, t, w) \leq_{\mathbf{s}} H_{\lambda \vee \mu}(\mathbf{x}, t, w) H_{\lambda \wedge \mu}(\mathbf{x}, t, w).$$



Guo



Kang



Liu

# Guo–Kang–Liu theorem

## Theorem (Guo–Kang–Liu 2026+)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$H_\lambda(\mathbf{x}, t, w) H_\mu(\mathbf{x}, t, w) \leq_s H_{\lambda \vee \mu}(\mathbf{x}, t, w) H_{\lambda \wedge \mu}(\mathbf{x}, t, w).$$

Hybrid Grothendieck polynomial  $H_\lambda(\mathbf{x}, t, w)$  is a unified generalization of Grothendieck and dual Grothendieck polynomials.

# Guo–Kang–Liu theorem

## Theorem (Guo–Kang–Liu 2026+)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$H_\lambda(\mathbf{x}, t, w) H_\mu(\mathbf{x}, t, w) \leq_s H_{\lambda \vee \mu}(\mathbf{x}, t, w) H_{\lambda \wedge \mu}(\mathbf{x}, t, w).$$

**ADS inequality:** appeared on arXiv on June 4 2026.

**Guo–Kang–Liu:** appeared on arXiv on July 3 2026.

# Guo-Kang-Liu theorem

## Theorem (Guo-Kang-Liu 2026+)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$H_\lambda(\mathbf{x}, t, w) H_\mu(\mathbf{x}, t, w) \leq_s H_{\lambda \vee \mu}(\mathbf{x}, t, w) H_{\lambda \wedge \mu}(\mathbf{x}, t, w).$$

**ADS inequality**: appeared on arXiv on June 4 2026.

**Guo-Kang-Liu**: appeared on arXiv on July 3 2026.

如此神速, 令人惊叹!

“Breathtaking in swiftness, wondrous to behold!”

**Open problem**

## Thomas–Yong conjecture

For symmetric polynomials  $f, g$ , we write

$$f \leq_{\tilde{\mathbf{g}}} g \iff \text{Expansion of } g - f \text{ in } \{\tilde{G}_\lambda(\mathbf{x})\} \\ \text{basis has nonnegative coefficients.}$$

### Conjecture (Thomas–Yong 2009)

*For any partitions  $\lambda$  and  $\mu$ ,*

$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_{\tilde{\mathbf{g}}} \tilde{G}_{\lambda \vee \mu}(\mathbf{x}) \tilde{G}_{\lambda \wedge \mu}(\mathbf{x}),$$

Mihalcea conjecture is a special case, as  
Grothendieck positivity implies Schur positivity.



感谢聆听!  
**THANK YOU!**



Preprint: [www.arxiv.org/abs/2606.06688](http://www.arxiv.org/abs/2606.06688)

Webpage: [www.math.rutgers.edu/~sc2518/](http://www.math.rutgers.edu/~sc2518/)