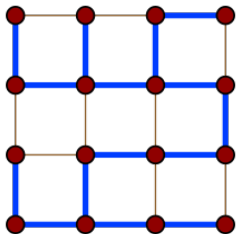


Spanning Trees and Continued Fractions

Swee Hong Chan

joint with Alex Kontorovich and Igor Pak



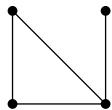
$$1 + \frac{4}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$$

What is a spanning tree?

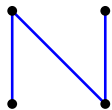
Let $G = (V, E)$ be a simple graph.

A **spanning tree** is a subset of edges of G that

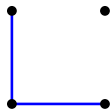
- includes all vertices (**spanning**),
- has no cycles (**tree**).



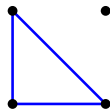
G



spanning tree

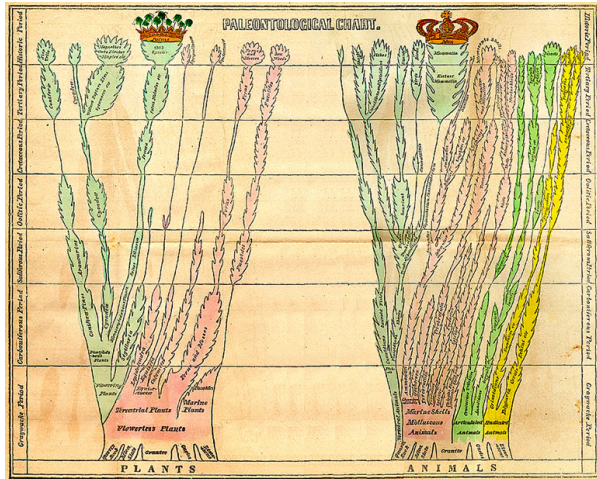


not spanning



not tree

Examples: Phylogenetic tree



From *Elementary Geology* (1840),
by Edward Hitchcock

How many spanning trees can a graph have?

Cayley's formula

Theorem (Borchardt 1860, Cayley 1889)

The number of spanning trees of a complete graph with n vertices is n^{n-2} .



Carl W. Borchardt

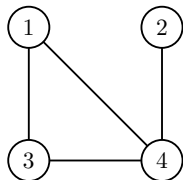


Arthur Cayley

Matrix tree theorem

Theorem (Kirchhoff 1847)

The number of spanning trees $t(G)$ of G is equal to the *determinant* of a minor of its *Laplacian matrix*.



G

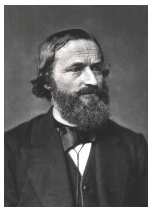
$$\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Laplacian

Matrix tree theorem

Theorem (Kirchhoff 1847)

*The number of spanning trees $t(G)$ of G is equal to the **determinant** of a minor of its **Laplacian matrix**.*



Gustav Kirchhoff

Note: Kirchhoff's paper has neither **matrices** nor **trees**.

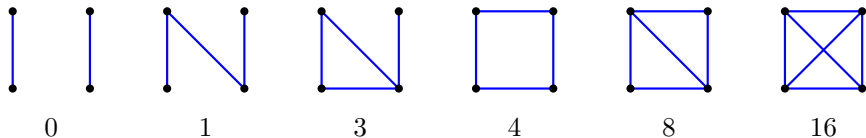
Sedláček's Problems

Set of spanning tree numbers

For $n \geq 1$, let

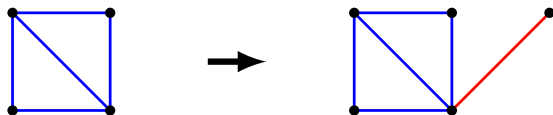
$$\mathcal{G}_n := \{\text{all simple graphs with } n \text{ vertices}\},$$

$$t(\mathcal{G}_n) := \left\{ \begin{array}{l} \text{number of spanning trees of all} \\ \text{simple graphs with } n \text{ vertices} \end{array} \right\}.$$

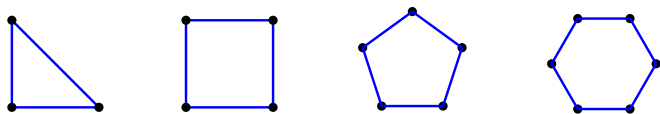


Properties of $t(\mathcal{G}_n)$

- We have $t(\mathcal{G}_n) \subseteq t(\mathcal{G}_{n+1})$.



- We have $\bigcup_{n \geq 1} t(\mathcal{G}_n) = \{0, 1, \cancel{2}, 3, 4, 5, \dots\}$.



Sedláček's First Problem

Problem (Sedláček 1966)

Describe the set of spanning tree numbers $t(\mathcal{G}_n)$.



O KOSTRÁCH KONEČNÝCH GRAFŮ

Jiří SEDLÁČEK, Praha

(Došlo dne 28. srpna 1965)

ON THE SPANNING TREES OF FINITE GRAPHS

Jiří SEDLÁČEK, Praha

Sedláček's First Problem

Theorem (Sedláček 1966)

For $n \geq 3$,

$$n^2 \leq |t(\mathcal{G}_n)| \leq n^{n-2}.$$

It is clear that the **lower bound** is not tight.

Conjecture

$$|t(\mathcal{G}_n)| \geq C^n \quad \text{for some } C > 1.$$

What was known

First **super-polynomial** lower bound
was due to **Azarija (2014)**:

$$|t(\mathcal{G}_n)| \geq e^{\Omega(\sqrt{n/\log n})}.$$

Best lower bound prior to our work
was due to **Stong (2022)**:

$$|t(\mathcal{G}_n)| \geq e^{\Omega(n^{2/3})}.$$

Sedláček's Second Problem

For $n \geq 1$, let

$\mathcal{P}_n := \{\text{all simple planar graphs with } n \text{ vertices}\}.$

Problem (Sedláček 1966)

Describe the set of spanning tree numbers of planar graphs $t(\mathcal{P}_n)$.

Note: Four-color theorem was proved in 1976.

What was known

It follows from [Euler's formula](#) that

$$|t(\mathcal{P}_n)| \leq 2^{|E|} \leq 8^n.$$

The best bounds prior to our work were:

$$e^{\Omega(n^{2/3})} \leq |t(\mathcal{P}_n)| \leq (5.2852)^n.$$

Upper bound was due to [Buchin-Schulz \(2010\)](#),
lower bound was due to [Stong \(2022\)](#).

First main result

Theorem 1 (C.–Kontorovich–Pak 2024+)

For sufficiently large n ,

$$|t(\mathcal{P}_n)| \geq (1.1103)^n.$$

Note that this implies

$$|t(\mathcal{G}_n)| \geq |t(\mathcal{P}_n)| \geq (1.1103)^n.$$

This is the first **exponential** lower bound for Sedláček's **First Problem**, and a **tight** lower bound for **Second Problem**.

Harder problem: Inverse counting problem

Input: Integer $T \geq 3$.

Problem: Construct graph G with $t(G) = T$ and

$$|V(G)| \leq c \log T \quad \text{for some } c > 0.$$

Solution to the above problem would imply,

$$|t(\mathcal{G}_n)| \geq (e^{1/c})^n,$$

and would solve [Sedláček's Problem](#).

Harder problem: Inverse counting problem

Input: Integer $T \geq 3$.

Problem: Construct graph G with $t(G) = T$ and

$$|V(G)| \leq c \log T \quad \text{for some } c > 0.$$

Can we solve this harder problem?

Harder problem: Inverse counting problem

Input: Integer $T \geq 3$.

Problem: Construct graph G with $t(G) = T$ and

$$|V(G)| \leq c \log T \quad \text{for some } c > 0.$$

Can we solve this harder problem?

Yes we (mostly) can!

Almost all integers

A set $S \subseteq \mathbb{N}$ contains **almost all integers** if

$$\lim_{N \rightarrow \infty} \frac{|S \cap \{1, \dots, N\}|}{N} = 1.$$

This is a weaker notion than requiring S to contain **all but finitely many integers**.

Second main result

Theorem 2 (C.–Kontorovich–Pak 2024+)

For *almost all integers* T , there exists a planar graph G with $t(G) = T$ and

$$|V(G)| \leq 56 \log_{\varphi} T,$$

where $\varphi := \frac{1+\sqrt{5}}{2} = 1.618$ is the *golden ratio*.

This implies the exponential lower bound

$$|t(\mathcal{P}_n)| \geq (\varphi^{1/56})^n \approx (1.008)^n,$$

which can be further improved to $(1.1103)^n$.

Connections to continued fractions

Continued fractions

For integers $a_1, \dots, a_k \geq 1$,

$$[a_1, \dots, a_k] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + a_k}}}.$$

Every rational number $\frac{t}{u} \leq 1$ can be written as a finite continued fraction using [Euclidean algorithm](#).

Furthermore, we have $k \leq \log_{\varphi} u$.

Connect spanning trees to continued fractions

Input: $b_1, \dots, b_\ell \geq 1$.

Output: Planar graph G and edge e with

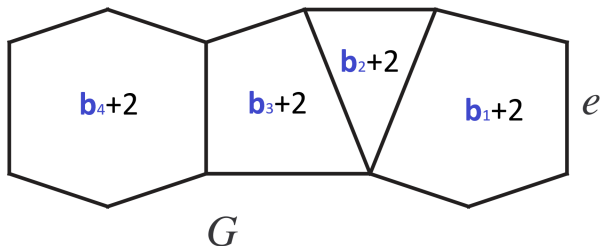
$$\frac{t(G - e)}{t(G/e)} = [b_1, 1, b_2, 1, \dots, b_\ell, 1],$$

$t(G - e)$ and $t(G/e)$ are coprime,

$$|V(G)| = b_1 + \dots + b_\ell + 2.$$

Here $G - e$ is graph deletion,
and G/e is graph contraction.

The silkworm graph



The i -th cycle has $b_i + 2$ vertices.

The example above has

$$[3, 1, 1, 1, 2, 1, 4, 1] = \frac{63}{229},$$

$$t(G - e) = 63, \quad t(G/e) = 229.$$

Zaremba's conjecture

Zaremba's conjecture

Conjecture (Zaremba 1972)

For every integer u , there exists coprime $t < u$ with

$$\frac{t}{u} = [a_1, \dots, a_k],$$

$$a_1, \dots, a_k \leq 5.$$

Conjecture is false if 5 is replaced with 4,
with $u = 54$.

Note $a_1 + \dots + a_k \leq 5 \log_{\varphi} u$
by Euclidean's algorithm.

Bourgain–Kontorovich theorem

Theorem (Bourgain–Kontorovich 2014)

For *almost all integers* u , that there exists coprime $t < u$ with

$$\frac{t}{u} = [a_1, \dots, a_k],$$

$$a_1, \dots, a_k \leq 50.$$

Huang (2015) has since improved
the bound from 50 to 5.

Bourgain–Kontorovich theorem

Theorem (Bourgain–Kontorovich 2014)

For *almost all integers* u , that there exists coprime $t < u$ with

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$$a_1, \dots, a_k \leq 50.$$

Huang (2015) has since improved
the bound from 50 to 5.

This is almost what we need for Sedláček's Problem.

Alternating BK theorem

Theorem (C.–Kontorovich–Pak 2024+)

For *almost all integers* t , there exists coprime $u > t$ with

$$\frac{t}{u} = [b_1, 1, b_2, 1, \dots, b_\ell, 1],$$

$$b_1, \dots, b_\ell \leq 110.$$

This is **exactly** what we need!

Back to inverse counting problem

Input: Integer $T \geq 3$.

Goal: Construct graph G with $t(G) = T$ and

$$|V(G)| \leq 56 \log_{\varphi} T.$$

We now give a construction that is **guaranteed** to work 99% of the time.

Solution to inverse counting problem

For $u \in \{T, \dots, 110T\}$ coprime to T :

- Compute continued fraction $\frac{T}{u} = [a_1, \dots, a_k]$.
- If $(a_1, \dots, a_k) = (b_1, 1, \dots, b_{k/2}, 1)$, $b_i \leq 110$:
construct silkworm $(b_1, \dots, b_{k/2})$ graph (G, e) .

Note that

$$t(G - e) = T,$$

$$|V(G)| = b_1 + \dots + b_{k/2} + 2 \leq 56 \log_{\varphi} T.$$

Output: graph $G - e$.

Solution to inverse counting problem

Input: Integer $T \geq 3$.

Goal: Construct graph G with $t(G) = T$ and

$$|V(G)| \leq 56 \log_{\varphi} T.$$

Alternating BK theorem thus guarantees our construction works 99% of the time...



Solution to inverse counting problem

Input: Integer $T \geq 3$.

Goal: Construct graph G with $t(G) = T$ and

$$|V(G)| \leq 56 \log_{\varphi} T.$$

Alternating BK theorem thus guarantees our construction works 99% of the time... and might still work for the other 1%.

Alternating Zaremba's conjecture

Conjecture

There exists an absolute constant $A > 0$, such that for *every* integer t , there exists coprime $u > t$ with

$$\frac{t}{u} = [b_1, 1, b_2, 1, \dots, b_\ell, 1],$$

$$b_1, \dots, b_\ell \leq A.$$

If this conjecture is true,
then our construction will **always** work.

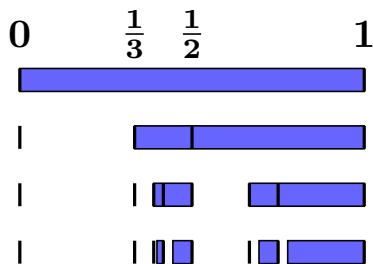
**Sketch of proof of
original and alternating BK theorem**

Cantor-like fractals

For $A \geq 2$,

$$\mathfrak{C}_A := \{ [a_1, a_2, \dots] \mid a_i \leq A \},$$

limit set of rational numbers in Zaremba's conjecture.



We would like to measure this set.

Hausdorff dimension

For $S \subseteq \mathbb{R}$ and $d \in \mathbb{R}_+$, Hausdorff measure $H^d(S)$ is

$$\liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} (b_i - a_i)^d : \bigcup_{i=1}^{\infty} (a_i, b_i) \supseteq S, b_i - a_i < \delta \right\}.$$

The Hausdorff dimension of S is

$$\text{Hdim}(S) := \inf \{ d \geq 0 : H^d(S) = 0 \}.$$

Note that, for Cantor-like fractals,

$$0 < \text{Hdim}(\mathfrak{C}_A) < 1, \quad \text{Hdim}(\mathfrak{C}_A) \nearrow 1 \text{ as } A \rightarrow \infty.$$

Black box: Orbital circle method

Theorem (Bourgain–Kontorovich 2014)

Let $A \geq 2$. Then, for *almost all integers* u , there exists coprime $t < u$ with

$$\frac{t}{u} = [a_1, \dots, a_k], \quad a_1, \dots, a_k \leq A$$

if

$$\text{Hdim}(\mathfrak{C}_A) > 0.984.$$

This reduces density one version of [Zaremba's conjecture](#) to computing [Hausdorff dimension](#).

Original BK theorem

Bourgain–Kontorovich (2014) computed that

$$\text{Hdim}(\mathfrak{C}_{50}) = 0.986\dots > 0.984.$$

Theorem (Bourgain–Kontorovich 2014)

For *almost all integers* u , there exists coprime $t < u$ with

$$\frac{t}{u} = [a_1, \dots, a_k], \quad a_1, \dots, a_k \leq 50.$$

Improvements have since been made on [orbital circle method](#) and [computing Hausdorff dimensions](#).

Improvements to orbital circle method

Frolenkov–Kan (2014) improves to

$$\text{Hdim}(\mathfrak{C}_A) > 0.83 \implies (\text{positive proportion}).$$

Huang (2015) improves to

$$\text{Hdim}(\mathfrak{C}_A) > 0.83 \implies (\text{density one}).$$

Kan (2015, 2017, 2021) improves to

$$\begin{aligned} \text{Hdim}(\mathfrak{C}_A) > 0.7749 &\implies (\text{positive proportion}) \\ &\implies (\text{density one}). \end{aligned}$$

Improvements to computing Hausdorff dimension

The [state of the art](#) algorithm to compute Hausdorff dimension is due to [Pollicott–Vytnova \(2022\)](#):

$$\text{Hdim}(\mathcal{C}_5) = 0.836829443680\dots$$

Recall the result of [Huang \(2015\)](#):

$$\text{Hdim}(\mathcal{C}_A) > 0.83 \implies (\text{density one}),$$

which gives the current best result for [Zaremba's conjecture](#).

Back to alternating BK theorem

For $A \geq 2$, the Hančl–Turek fractal is

$$\text{Hdim}(\mathfrak{D}_A) := \{ [b_1, 1, b_2, 1, \dots] \mid b_i \leq A \}.$$

Based on Kan (2021), we need to find A satisfying

$$\text{Hdim}(\mathfrak{D}_A) > 0.7749.$$

If such A exists, we get everything.

If such A does not exist, we get nothing.

Luck is on our side

Assisted by Pollicott–Vytnova and computers,

$$\text{Hdim}(\mathfrak{D}_{110}) = 0.7750\dots > 0.7749.$$

Theorem (C.–Kontorovich–Pak 2024+)

For *almost all integers* t , there exists coprime $u > t$ with

$$\frac{t}{u} = [b_1, 1, \dots, b_\ell, 1], \quad b_1, \dots, b_\ell \leq 110.$$

Pollicott (2025+) has since shown that

$$\text{Hdim}(\mathfrak{D}_{109}) = 0.774902739\dots > 0.7749.$$

Eureka!

Open problem

Improvement for Sedláček's First Problem

Conjecture

There exists $c > 0$ so that

$$|t(\mathcal{G}_n)| \geq 2^{cn \log n}.$$

Contrast this with the trivial upper bound

$$|t(\mathcal{G}_n)| \leq n^{n-2} \leq e^{n \log n}.$$

Solving this problem would most likely require new **graph constructions**.

Improvement for Sedláček's Second Problem

Alon–Bucić–Gishboliner (2025+) recently improved our lower bound from $(1.1103)^n$ to

$$|t(\mathcal{P}_n)| \geq (1.49)^n.$$

Problem

Does there exist $C > 1$ so that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |t(\mathcal{P}_n)| = C.$$

If C exists, then it must satisfy

$$1.49 < C < 5.2852.$$

THANK YOU!

Preprint: www.arxiv.org/abs/2411.18782

Webpage: www.math.rutgers.edu/~sc2518/

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