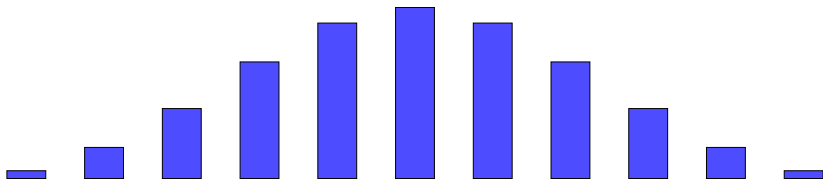


Complexity of Log-concave Inequalities for Matroids

Swee Hong Chan

joint with Igor Pak



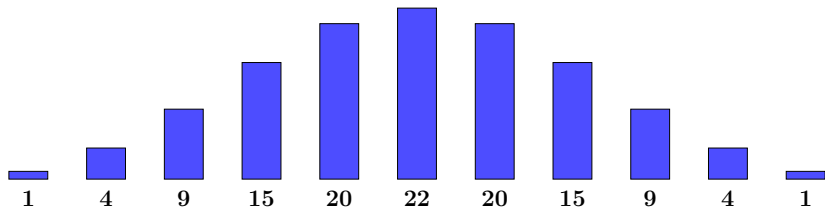
What is log-concavity?

A sequence $a_1, \dots, a_n \in \mathbb{N}_{\geq 0}$ is **log-concave** if

$$a_k^2 \geq a_{k+1} a_{k-1} \quad (1 < k < n).$$

Log-concavity (and positivity) implies **unimodality**:

$a_1 \leq \dots \leq a_m \geq \dots \geq a_n$ for some $1 \leq m \leq n$.



Log-concave shaped objects in real life



Esplanade – Theatres on the Bay,
Singapore (Credit TyLin).

Example 1: Binomial coefficients

$$a_k = \binom{n}{k} \quad k = 0, 1, \dots, n.$$

This sequence is **log-concave** because

$$\frac{a_k^2}{a_{k+1} a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1} \binom{n}{k-1}} = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example 2: Permutation inversion sequence

Let

$a_k :=$ number of $\pi \in S_n$ with k inversions,

where **inversion** of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is **log-concave** because

$$\sum_{0 \leq k \leq \binom{n}{2}} a_k q^k = [n]_q! = \prod_{i=1}^{n-1} (1 + q + q^2 + \dots + q^i)$$

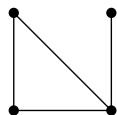
is a product of log-concave polynomials.

Example 3: Forests of a graph

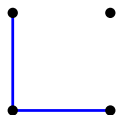
a_k = number of forests with k edges of graph G .

Forest is a subset of edges of G that has no cycles.

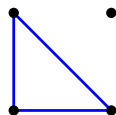
Log-concavity was conjectured for all **matroids** (Mason '72), and was proved through **combinatorial Hodge theory** (Huh '15).



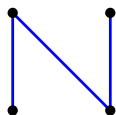
G



forest



not forest



spanning tree

Example 3: Forests of a graph

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Log-concavity was conjectured for all **matroids** (Mason '72), and was proved through **combinatorial Hodge theory** (Huh '15).



June Huh



Fields Medal

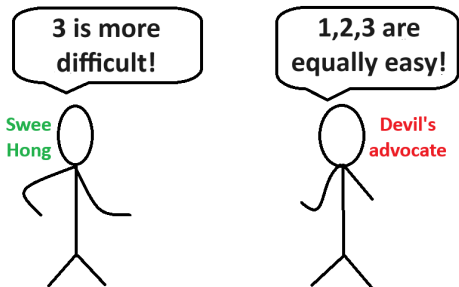
Motivation

Which log-concave inequality is more “difficult”?



Motivation

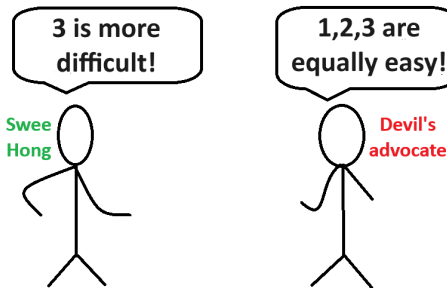
Which log-concave inequality is more “difficult”?



We will show that [REDACTED] (3)
is **strictly more** difficult than the rest,
using **Complexity Theory**.

Motivation

Which log-concave inequality is more “difficult”?



We will show that a **generalization** of (3) is **strictly more** difficult than the rest, using **Complexity Theory**.

Matroids

Object: Matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphic matroids

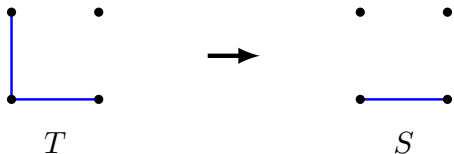
- X = edges of a graph G ,
- \mathcal{I} = forests in G .

Binary matroids

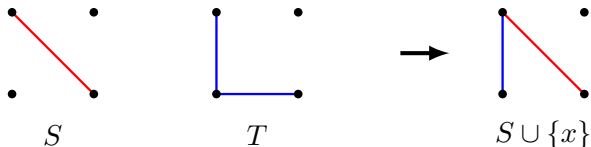
- X = set of vectors over finite field \mathbb{F}_2 ,
- \mathcal{I} = sets of linearly independent vectors.

Matroids: Axioms

- (Hereditary) If $S \subseteq T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$.



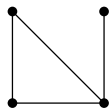
- (Exchange) If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



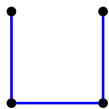
Matroid: Bases and ranks

A **basis** of \mathcal{M} is a **maximal** independent set.

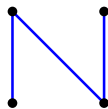
Rank r of \mathcal{M} is the size of the bases.



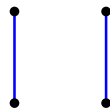
G



Basis 1



Basis 2



Not Basis

Matroid generalizes the notion of **vector spaces**.

Mason's conjecture

First Mason's conjecture

For matroid \mathcal{M} , let

$I(k) :=$ no. of **independents sets** with k elements.

For **graphic** matroid, $I(k)$ is no. of **forest** with k edges.

Conjecture (Mason '72)

The sequence $I(1), I(2), \dots$ is log-concave,

$$I(k)^2 \geq I(k+1)I(k-1) \quad (k \in \mathbb{N}),$$

First Mason's conjecture (continued)

Conjecture (Mason '72)

$$I(k)^2 \geq I(k+1)I(k-1) \quad (k \in \mathbb{N}).$$

Conjecture was proved for **graphic** matroids
by (**Huh '15**), and for **all** matroids
by (**Adiprasito–Huh–Katz '18**).

Both proofs used **combinatorial Hodge theory**.

First Mason's conjecture (continued)

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by (Huh '15), and for **all** matroids
by (Adiprasito–Huh–Katz '18).

Both proofs used **combinatorial Hodge theory**.

We will show that Mason's conjecture is
consequence of a **stronger inequality**.

Stanley–Yan inequality

Stanley–Yan inequality (simple case)

Let \mathcal{M} be a matroid with ground set X and rank r .

Fix a subset S of X . Let

$B(k) :=$ no. of **bases** B such that $|B \cap S| = k$,
multiplied by $r! \times \binom{r}{k}^{-1}$.

Theorem (Stanley '81, Yan '23)

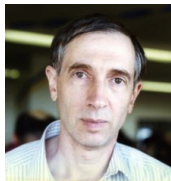
The sequence $B(1), B(2), \dots$ is log-concave,

$$B(k)^2 \geq B(k+1)B(k-1) \quad (k \in \mathbb{N}).$$

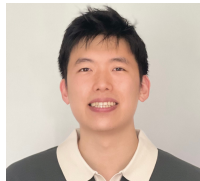
Stanley–Yan inequality (simple)

Theorem (Stanley '81, Yan '23)

$$B(k)^2 \geq B(k+1)B(k-1) \quad (k \in \mathbb{N}).$$



Richard Stanley



Alan Yan

Stanley–Yan inequality (simple)

Theorem (Stanley '81, Yan '23)

$$B(k)^2 \geq B(k+1)B(k-1) \quad (k \in \mathbb{N}).$$

Proved for **regular** matroids by (Stanley '81) using **Alexandrov–Fenchel inequality** for mixed volumes.

Proved for **all** matroids by (Yan '23) using theory of **Lorentzian polynomials**.

**Proof of Mason's conjecture
using Stanley–Yan inequality**

Proof of Mason's conjecture using SY inequality

Let

\mathcal{M} := original matroid in Mason's conjecture;

\mathcal{F} := matroid with r elements and with every subset being independent;

\mathcal{M}' := direct sum of \mathcal{M} and \mathcal{F} ;

S := ground set of \mathcal{M} .

Then

$$I(k) \text{ for } \mathcal{M} = \frac{1}{r!} \times B(k) \text{ for } \mathcal{M}'.$$

Proof of Mason's conjecture using SY inequality

Since

$$I(k) \text{ for } \mathcal{M} = \frac{1}{r!} \times B(k) \text{ for } \mathcal{M}',$$

we then conclude that

Stanley–Yan inequality for \mathcal{M}'
implies Mason's conjecture for \mathcal{M} .



Stanley–Yan inequality (full version)

Fix $d \geq 0$, disjoint subsets S, S_1, \dots, S_d of X ,
and $\ell_1, \dots, \ell_d \in \mathbb{N}$.

$B_d(k) :=$ number of bases B of \mathcal{M} such that
 $|B \cap S| = k, |B \cap S_i| = \ell_i$ for $i \in [d]$,
multiplied by $r! \times \binom{r}{k, \ell_1, \dots, \ell_d}^{-1}$.

Theorem (Stanley '81, Yan '23)

The sequence $B_d(1), B_d(2), \dots$ is log-concave,

$$B_d(k)^2 \geq B_d(k+1)B_d(k-1) \quad (k \in \mathbb{N}).$$

What we want to do

Theorem (Stanley '81, Yan '23)

The sequence $B_d(1), B_d(2), \dots$ is log-concave,

$$B_d(k)^2 \geq B_d(k+1)B_d(k-1) \quad (k \in \mathbb{N}).$$

Both LHS and RHS of this inequality has
combinatorial interpretations.

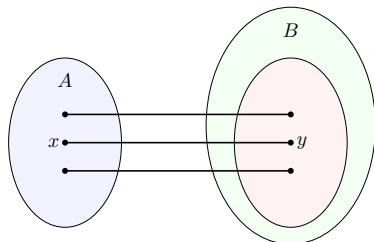
But we will show that this inequality has
no combinatorial injective proof.

Combinatorial injective proof

Combinatorial injection

An injection $f : A \rightarrow B$ is **combinatorial** if

- Given $x \in A$, the image $f(x)$ is computable in $\text{poly}(|x|)$ steps;
- Given $y \in B$, it takes $\text{poly}(|y|)$ steps **to decide if y is in image of f** ; and if so, the pre-image $f^{-1}(y)$ is computable in $\text{poly}(|y|)$ steps.



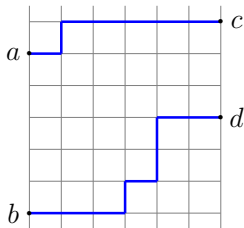
Example: Injective proof of binomial inequality

$$\binom{n}{k}^2 \geq \binom{n}{k+1} \binom{n}{k-1} \quad (1 < k < n).$$

This inequality has a **lattice path interpretation**:

$K(a \rightarrow c, b \rightarrow d) :=$ no. of pairs of north-east lattice paths from a to c and b to d ,

for $a, b, c, d \in \mathbb{Z}^2$.



Example: Injective proof of binomial inequality

Let

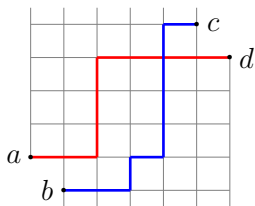
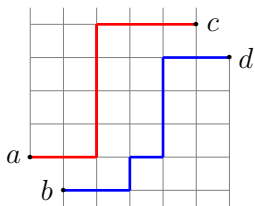
$$a = (0, 1), \quad c = (k, n - k + 1),$$

$$b = (1, 0), \quad d = (k + 1, n - k).$$

Then

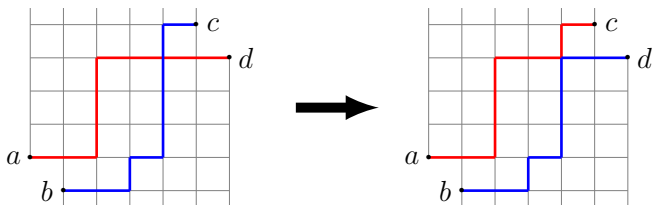
$$K(a \rightarrow c, b \rightarrow d) = \binom{n}{k},$$

$$K(a \rightarrow d, b \rightarrow c) = \binom{n}{k-1} \binom{n}{k+1}.$$



Example: Injective proof of binomial inequality

$f : K(a \rightarrow d, b \rightarrow c) \rightarrow K(a \rightarrow c, b \rightarrow d)$
is defined by **path-swapping injections**.



Images of f are pairs of lattice paths that **intersects**.

First main result

Theorem 1 (C.–Pak '24+)

There is **no combinatorial injective proof** for the Stanley–Yan inequality, assuming *polynomial hierarchy does not collapse*.

The assumption above is slightly stronger than $P \neq NP$, and is widely used in Complexity Theory.

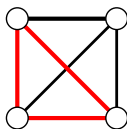
Polynomial hierarchy

Level 0: Complexity class P

P := Decision problems that, given input x , can be solved in $\text{poly}(|x|)$ time.

Example (Problem in P)

Does a graph G contain a triangle?



This complexity class is denoted by Σ_0^P .

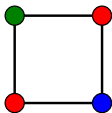
Level 1: Complexity class NP

Problems asking about **existence** of

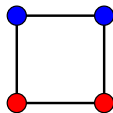
NP := a solution S for input x , where **validity** of S can be **verified** in $\text{poly}(|x|)$ time.

Example (Problem in NP)

Does a graph G have a proper 3-coloring?



Proper coloring



Improper coloring

This complexity class is denoted by Σ_1^P .

Oracle machine

An **oracle machine** is a black box capable of solving problems from a **given class** in a single operation.



Level i of polynomial hierarchy

The class $\Sigma_i^P := \text{NP}^{\Sigma_{i-1}^P}$ is

Problems asking about existence of a solution S for input x , where validity of S can be verified in $\text{poly}(|x|)$ time, augmented by Σ_{i-1}^P -oracle.

Note that

$$\Sigma_0^P \subseteq \Sigma_1^P \subseteq \Sigma_2^P \subseteq \Sigma_3^P \subseteq \dots$$

Polynomial hierarchy (PH)

Polynomial hierarchy is the union of all Σ_i^P 's,

$$\text{PH} := \bigcup_{i=0}^{\infty} \Sigma_i^P.$$

Conjecture

Polynomial hierarchy does not collapse,

$$\Sigma_0^P \subsetneq \Sigma_1^P \subsetneq \Sigma_2^P \subsetneq \Sigma_3^P \subsetneq \dots$$

- $\Sigma_0^P \neq \Sigma_1^P$ is equivalent to $P \neq NP$.
- $\Sigma_1^P \neq \Sigma_2^P$ is equivalent to $NP \neq \text{coNP}$.

Back to the main result

Theorem (C.–Pak '24+)

There is **no combinatorial injective proof** for the Stanley–Yan inequality, assuming $\Sigma_2^P \neq \Sigma_3^P$.

Proof ideas

Ingredient 1: Study equality conditions

Let **SY-Equal** be the decision problem:

Input: Binary matroid \mathcal{M} , subsets S, S_1, \dots, S_d ,
integers k, ℓ_1, \dots, ℓ_d .

Output: YES if $B_d(k)^2 = B_d(k+1) B_d(k-1)$.
NO if $B_d(k)^2 > B_d(k+1) B_d(k-1)$.

Understanding **complexity of equality conditions** is
key to showing **combinatorial injections** do not exist.

Equality conditions vs combinatorial injections

Suppose that combinatorial injection existed:

$$f : B_d(k+1) \times B_d(k-1) \longrightarrow B_d(k)^2.$$

Then, given $y \in \text{RHS}$, it would take $\text{poly}(|y|)$ time to **verify** if y belongs to image of f .

This would imply $\text{SY-Equal} \in \text{coNP}$.

Problem reduces to showing $\text{SY-Equal} \notin \text{coNP}$.

Ingredient 2: Reduce problem to counting bases

Let **#Bases** be the counting problem:

Input: Binary matroid \mathcal{M} .

Output: Number of bases of \mathcal{M} .

Lemma (C.–Pak 24+)

*There exists a **nondeterministic polynomial-time Turing reduction** from **#Bases** to **SY-Equal**.*

Strategy: show that **#Bases** is ‘difficult’, then use Lemma to imply **SY-Equal** is also “difficult”.

Complexity class #P

Problems asking about **existence** of
NP := a solution S for input x , where validity of S can be verified in polynomial time.

Problems asking for **number** of solutions
#P := S for input x , where validity of S can be verified in polynomial time.

Example (Problem in #P)

Count the number of proper 3-colorings of graph G .

Complexity class #P: Equivalent definition

A problem is in #P if, for any input x ,

$$\text{Output} = \sum_{S \in \{0,1\}^{\text{poly}(|x|)}} V(x, S)$$

where

$$V(x, S) \in \{0, 1\}$$

can be evaluated in $\text{poly}(|x|)$ time.

Note that the size of the **output** is at most exponential relative to the input x .

Ingredient 3: Complexity of #Bases

Theorem (Knapp–Noble '23+)

#Bases is #P-complete for binary matroids.

We would like to use the complexity of #Bases to determine the complexity of SY-Equal.

Ingredient 4: Toda's Theorem

Theorem (Toda '91)

Every problem in PH has a polynomial-time Turing reduction to a problem in #P, i.e.

$$\text{PH} \subseteq \text{P}^{\#\text{P}}.$$

Theorem allows us to connect complexity of **decision problems** to complexity of **counting problems**.

Combine all the ingredients

Start with Toda's Theorem:

$$PH \subseteq P^{\#P}.$$

Since $\#Bases$ is $\#P$ -complete:

$$PH \subseteq P^{\#Bases}.$$

Now reduce $\#Bases$ to $SY=Equal$:

$$PH \subseteq NP^{SY=Equal}.$$

Combine all the ingredients

Now suppose that **combinatorial injection** existed.

Then **SY-Equal** belongs to **coNP**:

$$\text{PH} \subseteq \text{NP}^{\text{SY-Equal}} \subseteq \text{NP}^{\Sigma_1^{\text{P}}} = \Sigma_2^{\text{P}} .$$

Thus **PH** would collapse to the second level.



Combine all the ingredients

Now suppose that **combinatorial injection** existed.

Then **SY-Equal** belongs to **coNP**:

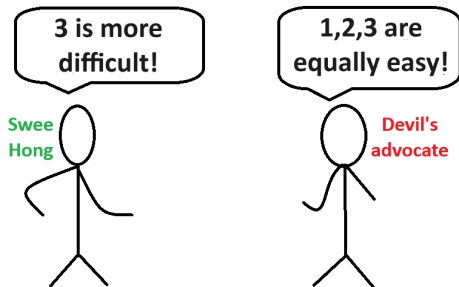
$$\text{PH} \subseteq \text{NP}^{\text{SY-Equal}} \subseteq \text{NP}^{\Sigma_1^{\text{P}}} = \Sigma_2^{\text{P}} .$$

Thus **PH** would collapse to the second level.

Theorem (C.-Pak '24+)

No combinatorial injective proof for *Stanley–Yan inequality* for binary matroids, assuming $\Sigma_2^{\text{P}} \neq \Sigma_3^{\text{P}}$.

Recall our goal



We will now show that **Stanley–Yan inequality** is strictly more difficult than the **binomial inequality** and **permutation inversion inequality**.

Second main result

Consider the following computational problem:

Input: Binary matroid \mathcal{M} , subsets S, S_1, \dots, S_d ,
integers k, ℓ_1, \dots, ℓ_d .

Output: $B_d(k)^2 - B_d(k+1) B_d(k-1)$.

Theorem 2 (C.-Pak '24+)

The problem above **does not belong to #P**,
assuming $\Sigma_2^P \neq \Sigma_3^P$.

Second main result

Theorem (C.–Pak '24+)

The problem of computing

$$B_d(k)^2 - B_d(k+1) B_d(k-1)$$

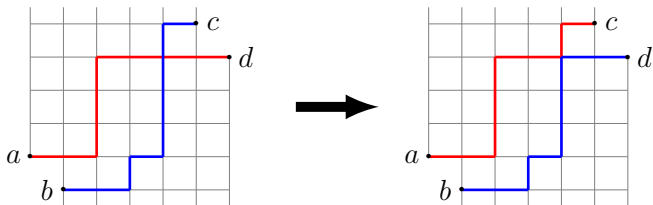
is **not in #P**, assuming $\Sigma_2^P \neq \Sigma_3^P$.

Both LHS and RHS of Stanley–Yan inequality belongs to **#P**, but their difference **does not**.

Example 1: Binomial inequality

It follows from **path-swapping injections** that

$\binom{n}{k}^2 - \binom{n}{k+1} \binom{n}{k-1} =$ number of **non-intersecting lattice paths** from a to c and b to d .



Thus the defect of this inequality belongs to **#P**.

Example 2: Permutation inversion inequality

Let a_k = number of $\pi \in S_n$ with k inversions.

$$\text{Then } \sum_{0 \leq k \leq \binom{n}{2}} a_k q^k = \prod_{i=1}^{n-1} (1 + q + \dots + q^i)$$

is computable in $\text{poly}(n)$ time.

Thus $a_k^2 - a_{k+1}a_{k-1}$ is computable in $\text{poly}(n)$ time;

and thus belongs to $\#P$.

Conclusion

We compare three log-concave inequalities:

Binomial inequality: **in #P**;

Permutation inversion inequality: **in #P**;

Stanley–Yan inequality: **not in #P**.

This differentiates **Stanley–Yan inequality**
from **binomial inequality** and **permutation
inversion inequality**.

Open Problem

Conjecture

Defect of Mason's conjecture

$$I(k)^2 - I(k+1)I(k-1) \notin \#P.$$

We have shown defect of **Stanley–Yan inequality** does not belong to #P, but not **Mason's conjecture**.

THANK YOU!

Paper: www.arxiv.org/abs/2309.05764

www.arxiv.org/abs/2407.19608

Webpage: www.math.rutgers.edu/~sc2518/

Email: sweehong.chan@rutgers.edu