In between random walk and rotor walk

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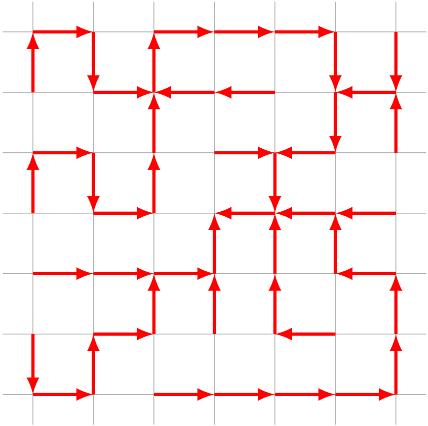
Joint work with Lila Greco, Lionel Levine, Boyao Li



Simple random walk on $\mathbb{Z}^2$

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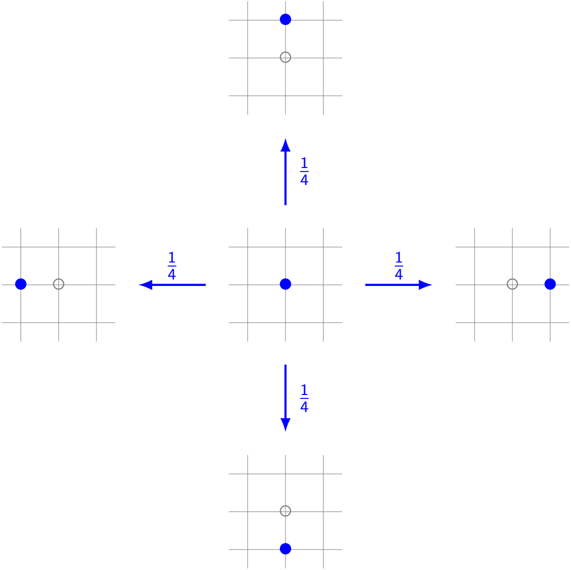


Simple random walk on $\mathbb{Z}^2$

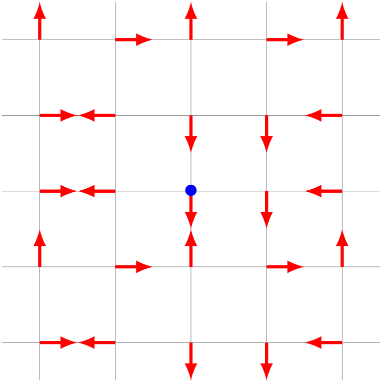
- Visits every site infinitely often? Yes!
- Scaling limit? The standard 2-D Brownian motion, i.e.

$$\frac{1}{\sqrt{n}} \underbrace{X_{[nt]}}_{\substack{\text{Location of the}\\ \text{walker at time } [nt]}} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}} \underbrace{(B_1(t), B_2(t))}_{\substack{\text{Independent standard}\\ \text{Brownian motions}}} \qquad t \geq 0.$$

Rotor walk on $\mathbb{Z}^2$

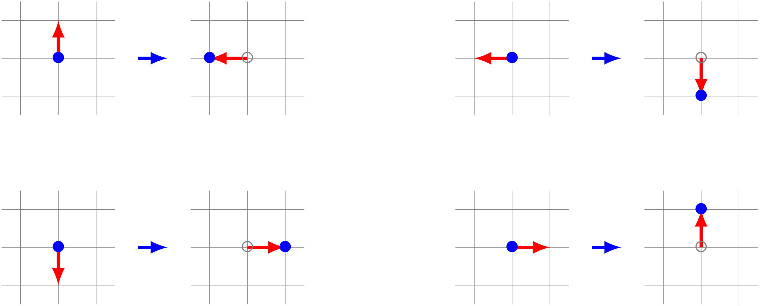
Rotor walk on $\mathbb{Z}^2$

Put a signpost at each site.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



The signpost says:
"This is the way you went the last time you were here",
(assuming you ever were!)

Conjectures for a rotor walk on $\mathbb{Z}^2$

If the initial signposts are i.i.d uniform among the four directions, then

- (PDDK '96) Visits every site infinitely often?
- (Kapri-Dhar '09) Scaling limit? The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc (!)

More randomness please!

Well studied

Many open problems

Random

Deterministic

More randomness please!

Well studied

Let's study this!!!

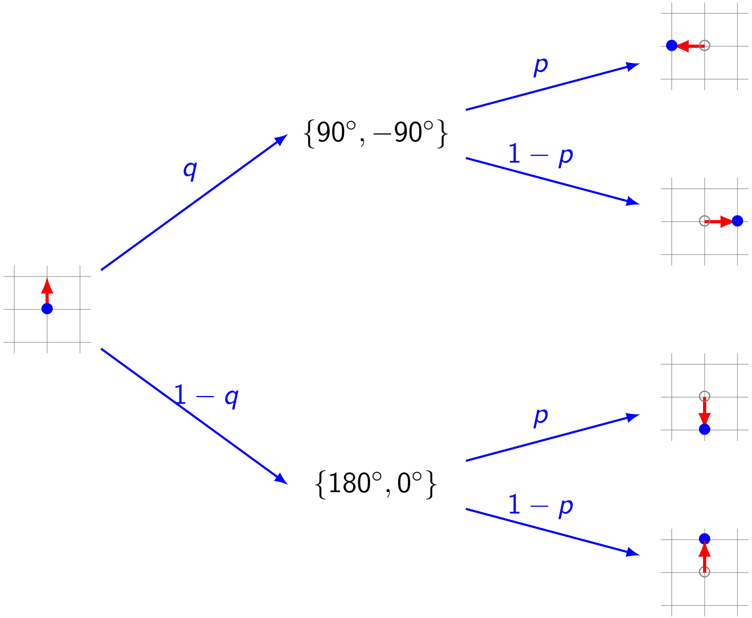
Many open problems

Random

Deterministic

p,q-rotor walk on $\mathbb{Z}^2$

p,q-rotor walk on $\mathbb{Z}^2$



Scaling limit for p,q-rotor walk on $\mathbb{Z}$

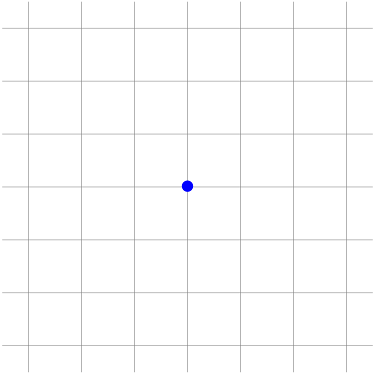
(Huss, Levine, Sava-Huss 16+) The scaling limit for p,q-rotor walk on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$, i.e.

$$Y(t) = \underbrace{B(t)}_{\substack{\text{Standard}\\ \text{Brownian}\\ \text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{Perturbation when}\\ \text{hitting maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{Perturbation when}\\ \text{hitting minimum}}}, \qquad t \geq 0,$$

Scaling limit for p,q-rotor walk on $\mathbb{Z}^2$

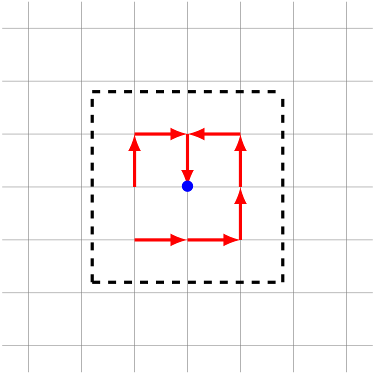
- Conjecture: the scaling limit for p,q-rotor walk on $\mathbb{Z}^2$ is a "2-D perturbed Brownian motion (?)".
- Problem: What is "2-D perturbed Brownian motion(?)".
- Weaker conjecture: the scaling limit for p,q-rotor walk on $\mathbb{Z}^2$ when $p = \frac{1}{2}$ is the standard 2-D Brownian motion.

Uniform spanning tree plus one edge (USTP)



From the perspective of the walker, this distribution is a stationary distribution for p,q-rotor walk on $\mathbb{Z}^2$.

Uniform spanning tree plus one edge (USTP)



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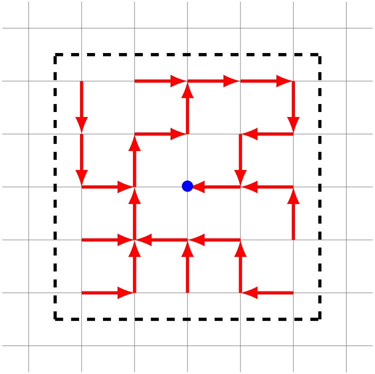

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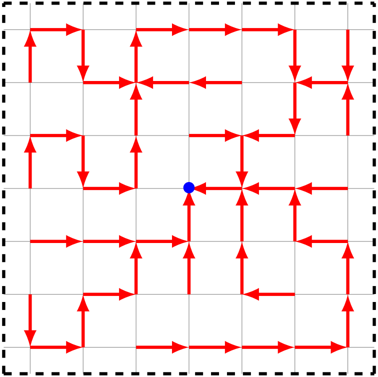

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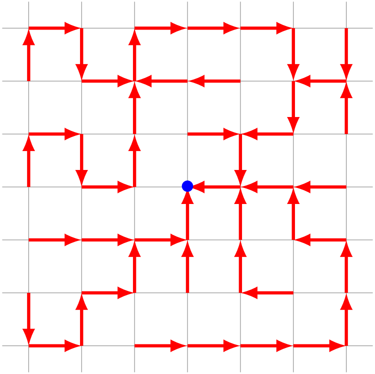

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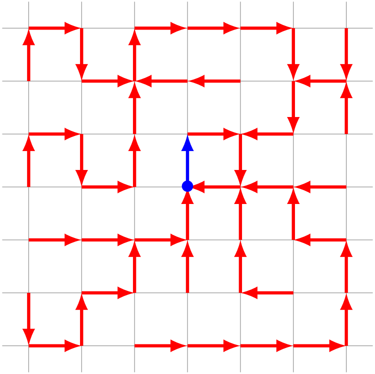

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Scaling limit for p,q-rotor walk on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '17+)

Let $p = \frac{1}{2}$, $q \in (0,1)$, and let the uniform spanning tree plus one edge be the initial signpost. Then with probability 1 the p,q-rotor walk on $\mathbb{Z}^2$ scales to the standard 2-D Brownian motion, i.e.

$$\frac{1}{\sqrt{n}} \underbrace{X_{[nt]}}_{\substack{\text{Location of the} \\ \text{walker at time } [nt]}} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}} \underbrace{(B_1(t), B_2(t))}_{\substack{\text{Independent} \\ \text{Brownian motions}}} \qquad t \geq 0.$$

The tools that we use include Martingale central limit theorem and pointwise ergodic theorem.

What is next?

- Scaling limit for when $p \neq \frac{1}{2}$?
  - Need to define the “2-D perturbed Brownian motion (?)”.
- Scaling limit for higher dimensional lattice $\mathbb{Z}^d$?
  - Yes for $d \in \{3, 4\}$.
  - Open for $d \geq 5$.
- Rotor walks with USTP as initial signposts?
  - Does NOT visit every site infinitely often (Florescu, Levine, Peres ‘16).
  - Scaling limit?

THANK YOU!

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