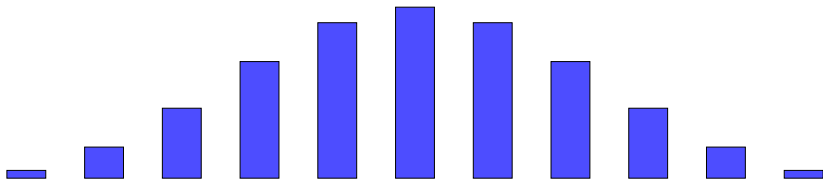


Log-concavity, Cross Product Conjectures, and FKG Inequalities in Order Theory

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joint with Igor Pak and Greta Panova



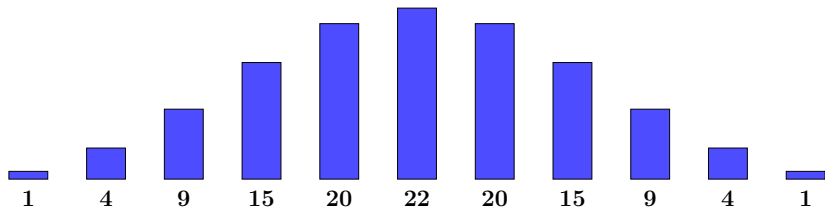
What is log-concavity?

A sequence $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ is **log-concave** if

$$a_k^2 \geq a_{k+1} a_{k-1} \quad \text{for all } 1 < k < n.$$

Log-concavity (and positivity) implies **unimodality**:

$$a_1 \leq \dots \leq a_m \geq \dots \geq a_n \quad \text{for some } 1 \leq m \leq n.$$



Example: binomial coefficients

$$a_k = \binom{n}{k} \quad k = 0, 1, \dots, n.$$

This sequence is **log-concave** because

$$\frac{a_k^2}{a_{k+1} a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1} \binom{n}{k-1}} = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

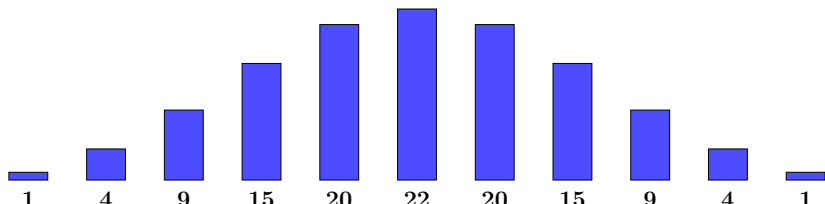
a_k = number of $\pi \in S_n$ with k inversions,

where **inversion** of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is **log-concave** because

$$\sum_{0 \leq k \leq \binom{n}{2}} a_k q^k = [n]_q! = \prod_{i=1}^{n-1} (1 + q + \dots + q^i)$$

is a product of log-concave polynomials.

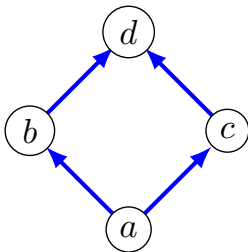


Log-concavity is a **widespread** phenomenon observed in **numerous** subjects in mathematics.

Today we focus on log-concavity in **posets**.

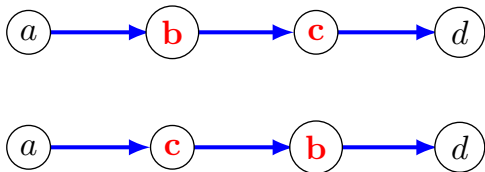
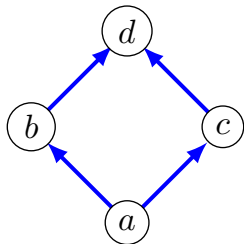
Partially ordered sets

A poset P is a set X with a partial order \prec on X .



Linear extension

A linear extension L is a complete order of \prec .



We write $L(x) = k$ if x is k -th smallest in L .

Stanley's inequality

Fix $z \in P$.

N_k is probability that $\mathcal{L}(z) = k$,

where \mathcal{L} is uniform random linear extension of P .

Theorem (Stanley '81)

For every poset and $k \geq 1$,

$$N_k^2 \geq N_{k+1} N_{k-1}.$$

The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Our contribution

Problem

(Folklore, Graham '83, Biró-Trotter '11, Stanley '14)

Give a *combinatorial proof* of Stanley's inequality.

Answer (C.-Pak '21+)

More *combinatorial proof* for Stanley's inequality, with generalizations to weighted version.

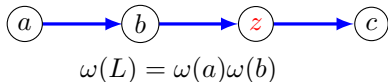
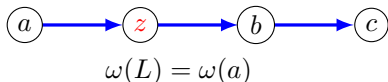
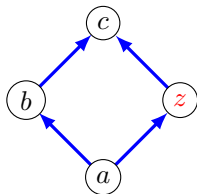
Order-reversing weight

A weight $\omega : X \rightarrow \mathbb{R}_{>0}$ is **order-reversing** if

$$\omega(x) \geq \omega(y) \quad \text{whenever} \quad x \prec y.$$

Weight of linear extension L is

$$\omega(L) := \prod_{L(x) < L(z)} \omega(x).$$



Weighted Stanley's inequality

Let $N_{\omega,k}$ be probability that $\mathcal{L}(z) = k$,
where \mathcal{L}_ω is ω -weighted random linear extension.

Theorem 1 (C.-Pak '21+)

For every poset and $k \geq 1$,

$$N_{\omega,k}^2 \geq N_{\omega,k+1} N_{\omega,k-1}.$$

Proof used **combinatorial atlas** method,
a new tool to establish log-concave inequalities.

Applications of log-concavity

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For finite poset that is not completely ordered, there exist elements x, y :

$$\frac{1}{3} \leq \mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] \leq \frac{2}{3},$$

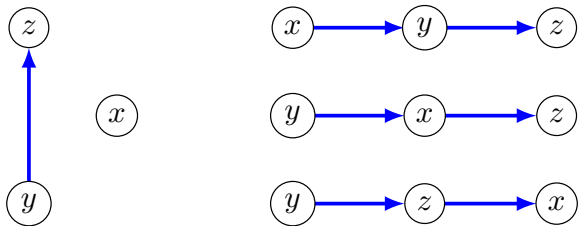
where \mathcal{L} is uniform random linear extension of P .

Quote (Brightwell-Felsner-Trotter '95)

*“This problem remains one of the **most intriguing problems** in the combinatorial theory of posets.”*

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper, lower bound are achieved by this poset:



$$\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] = \frac{1}{3}; \quad \mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)] = \frac{2}{3}.$$

Evidence for $\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture was verified for

- Posets of width 2 (Linial '84),
- Posets of at most 11 elements (Peczarki '06)
- Young diagram posets (Olson–Sagan '18)
- Many other posets ...

The big breakthrough

Theorem (Kahn-Saks '84)

For poset that is not completely ordered, there exist elements x, y :

$$\frac{3}{11} \leq \mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] \leq \frac{8}{11},$$

roughly between 0.273 and 0.727.

Proof used [log-concavity](#) as a crucial component.

Proof sketch of Kahn-Saks Theorem

Find $x, y \in P$ such that

$$|h(y) - h(x)| \leq 1,$$

where $h(x) := \mathbb{E}[\mathcal{L}(x)]$ and $h(y) := \mathbb{E}[\mathcal{L}(y)]$.

Let F_k be probability that $\mathcal{L}(y) - \mathcal{L}(x) = k$.

$$\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] = F_1 + F_2 + \cdots + F_n,$$

$$\mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)] = F_{-1} + F_{-2} + \cdots + F_{-n}.$$

Proof sketch of Kahn-Saks Theorem

Since $|h(y) - h(x)|$ is small,

$$F_1 + 2F_2 + \cdots + nF_n \approx F_{-1} + 2F_{-2} + \cdots + nF_{-n}.$$

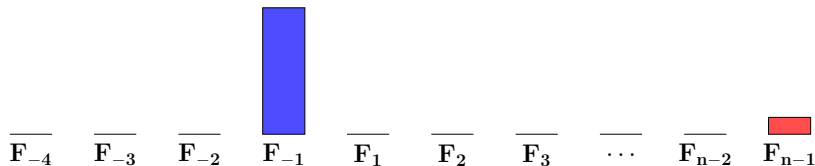
One can hope this implies

$$F_1 + F_2 + \cdots + F_n \approx F_{-1} + F_{-2} + \cdots + F_{-n},$$

which would then imply

$$\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] \approx \mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)] \approx 0.5.$$

But things can go really wrong:



Log-concavity comes to rescue

Theorem (Kahn–Saks '84)

For $k \neq 0$,

$$F_k^2 \geq F_{k+1} F_{k-1},$$

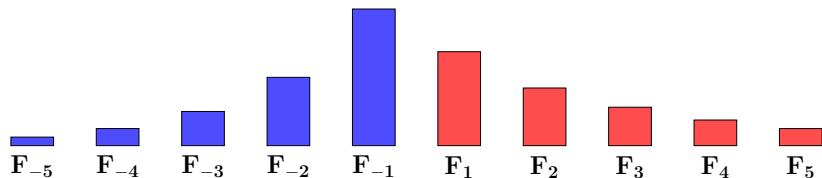
$$F_{-k}^2 \geq F_{-(k+1)} F_{-(k-1)}.$$

This generalizes [Stanley's inequality](#), and was proved by [Aleksandrov-Fenchel inequality](#).

Proof sketch of Kahn-Saks Theorem

Log-concavity (and other ineqs.) imply:

- $\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)]$ is maximized (resp. minimized) when F_1, F_2, \dots, F_n is geometric sequence,
- $\mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)]$ is minimized (resp. maximized) when $F_{-1}, F_{-2}, \dots, F_{-n}$ is geometric sequence.



Combined with $|h(y) - h(x)| \leq 1$, the result follows.

Best known bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture

Theorem (Brightwell-Felsner-Trotter '95)

For poset that is not completely ordered, there exist elements x, y :

$$\frac{5 - \sqrt{5}}{10} \leq \mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] \leq \frac{5 + \sqrt{5}}{10},$$

roughly between 0.276 and 0.724.

Note: Kahn–Saks bound was 0.273 and 0.727.

This bound cannot be improved for [infinite posets](#).

Cross Product Conjecture

New ingredient: Cross Product Conjecture

Fix $x, y, z \in P$. Let $F(k, \ell)$ be probability that

$$\mathcal{L}(y) - \mathcal{L}(x) = k \quad \text{and} \quad \mathcal{L}(z) - \mathcal{L}(y) = \ell.$$

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k, \ell) F(k + 1, \ell + 1) \leq F(k + 1, \ell) F(k, \ell + 1).$$

Equivalently,

$$\det \begin{bmatrix} F(k, \ell) & F(k, \ell + 1) \\ F(k + 1, \ell) & F(k + 1, \ell + 1) \end{bmatrix} \leq 0.$$

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k, \ell) F(k + 1, \ell + 1) \leq F(k + 1, \ell) F(k, \ell + 1).$$

Brightwell-Felsner-Trotter proved the case $k = \ell = 1$ by Ahlswede–Daykin inequality.

Combined with Kahn–Saks proof, this gives the $\frac{5 \pm \sqrt{5}}{10}$ bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture.

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, l \geq 1$,

$$F(k, l) F(k + 1, l + 1) \leq F(k + 1, l) F(k, l + 1).$$

Quote (Brightwell-Felsner-Trotter '95)

“Something more powerful seems to be needed to prove general form of Cross Product Conjecture.”

Our results

Theorem 2 (C.-Pak-Panova '22)

Cross Product Conjecture is true for posets of width two.

Proved algebraically using **matrix algebra** argument and combinatorially through **Lindström–Gessel–Viennot** type argument.

Our results

Theorem 3 (C.-Pak-Panova '23+)

For every poset and $k, \ell \geq 1$,

$$F(k, \ell) F(k + 1, \ell + 1) < 2F(k + 1, \ell) F(k, \ell + 1).$$

Proof is based on [Favard's inequality](#) for mixed volumes, for which factor of 2 is tight for general geometric objects.

On the other hand, for specific classes of posets this factor of 2 [can be improved](#).

A new protagonist

We now shift the attention
from **linear extensions** to **order-preserving maps**.

Order-preserving maps

Fix poset $P = (X, \prec)$.

A map $M : X \rightarrow \{1, \dots, t\}$ is **order-preserving** if

$$x \prec y \quad \text{implies} \quad M(x) \leq M(y).$$

Linear extensions are order-preserving maps that are also bijections to $\{1, \dots, |X|\}$.

Previously on linear extensions ...

- Log-concavity?
Solved: Stanley '81, Kahn–Saks '84, C.-Pak
- Cross-product conjecture?
Open: Brightwell–Felsner–Trotter '95, C.-Pak-Panova '22
- $\frac{1}{3}$ – $\frac{2}{3}$ Conjecture?
Open: Kahn–Saks '84, Brightwell–Felsner–Trotter '95

Can we **improve** on these results
for **order-preserving maps**?

Log-concavity for order-preserving maps

Graham's conjecture

Fix $z \in P$ and positive integer t .

G_k is probability that $\mathcal{M}(z) = k$,

where \mathcal{M} is uniform random ord.-pres. map $X \rightarrow [t]$.

Conjecture (Graham '83)

For every poset and $k \geq 1$,

$$G_k^2 \geq G_{k+1} G_{k-1}.$$

Graham's conjecture

Quote (Graham '83)

*"It would seem that [the conjecture] should have a proof based on the **FKG or AD inequalities**.*

However, such a proof has up to now successfully eluded all attempts to find it".

What is Harris/FKG/AD inequalities?

They are fundamental inequalities in probability that shows, in many random systems, increasing events are positively correlated.

Example

For any $a, b, c, d \in V$ in Erdős–Renyi random graph,

$$\mathbb{P}[a \leftrightarrow b, c \leftrightarrow d] \geq \mathbb{P}[a \leftrightarrow b] \mathbb{P}[c \leftrightarrow d],$$

where $a \leftrightarrow b$ is event that a and b are connected.

Presence of one path **increases** probability of other path.

Graham's conjecture is true

Theorem (Daykin–Daykin–Paterson '84)

For every poset and $k \geq 1$,

$$G_k^2 \geq G_{k+1} G_{k-1}.$$

Proof used an explicit **injective** argument,
not based on FKG/AD inequality.

Quote (Daykin–Daykin–Paterson '84)

“[Proof using FKG or Ahlswede–Daykin inequality]
have as yet **eluded discovery**”.

Our results

Theorem 4 (C.–Pak '22+)

New proof of Daykin–Daykin–Paterson inequality based on Ahlswede–Daykin inequality, with generalization to multi-weighted version.

This proof validates Graham's prediction.

**Cross product conjecture for
order-preserving maps**

Our results

Fix $x, y, z \in P$ and $t \geq 1$. Let $G(k, \ell)$ be probability

$$\mathcal{M}(y) - \mathcal{M}(x) = k \quad \text{and} \quad \mathcal{M}(z) - \mathcal{M}(y) = \ell,$$

where \mathcal{M} is uniform random ord.-pres. map $X \rightarrow [t]$.

Theorem 5 (C.-Pak '22+)

For all integers k, ℓ ,

$$G(k, \ell) G(k + 1, \ell + 1) \leq G(k + 1, \ell) G(k, \ell + 1).$$

This proves **cross product conjecture** for
order-preserving maps.

Our results

Theorem (C.–Pak '22+)

For all integers k, ℓ ,

$$G(k, \ell) G(k + 1, \ell + 1) \leq G(k + 1, \ell) G(k, \ell + 1).$$

Proof is based on same approach discovered when proving Daykin–Daykin–Paterson inequality.

This approach does not work for **linear extensions**, where inequality is known with factor of 2 in RHS.

$\frac{1}{3}$ – $\frac{2}{3}$ Conjecture for order-preserving maps

Conjecture

For finite poset that is not completely ordered, there exist elements x, y :

$$\frac{1}{3} \leq \lim_{t \rightarrow \infty} \mathbb{P}[\mathcal{M}_t(x) < \mathcal{M}_t(y)] \leq \frac{2}{3},$$

where \mathcal{M}_t is uniform random o.p. map $X \rightarrow [t]$.

This is in fact equivalent to $\frac{1}{3}$ – $\frac{2}{3}$ Conjecture for linear extensions.

All recent advances unfortunately do not improve known bounds for this conjecture.

Open problem

Kahn-Saks Conjecture

$\delta(P)$ is largest number such that there exist $x, y \in P$:

$$\delta(P) \leq \mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)] \leq 1 - \delta(P).$$

Note that $\frac{1}{3} - \frac{2}{3}$ Conjecture is equivalent to

$\delta(P) \geq \frac{1}{3}$ for P not completely ordered.

Conjecture (Kahn-Saks '84)

$$\delta(P) \rightarrow \frac{1}{2} \quad \text{as} \quad \text{width}(P) \rightarrow \infty.$$

Kahn-Saks Conjecture

Conjecture (Kahn-Saks '84)

$$\delta(P) \rightarrow \frac{1}{2} \quad \text{as} \quad \text{width}(P) \rightarrow \infty.$$

Komlós '90 proved Conjecture for posets with $\Omega\left(\frac{n}{\log \log \log n}\right)$ minimal elements.

C.-Pak-Panova '21 proved Conjecture for Young diagram posets with fixed width.

THANK YOU!

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