Stability Phenomenon for Quadratic Permanent Inequality

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joint with Igor Pak

The Great Pancake Making Competition

Contestants are asked to make perfect pancakes.

Perfect pancake is one that minimizes the crust (perimeter) for a given amount of batter (area).

The perfect pancake

Claim (Zenodorus, 2nd century BCE)

The perfect pancake must be circle-shaped.

The claim was known since antiquity, but a fully rigorous proof did not appear until 19th century.

Isoperimetric problem (dimension 2)

Theorem (Isoperimetric inequality)

For a closed 2-D region,

$$P^2 \geq 4\pi A,$$

where P is perimeter and A is area of the region.

Theorem (Equality cases)

Equality occurs $\iff$ the region is a circle.

Historically credited to Steiner (1838) for intuition, and Weierstrass (1879) for rigorous proof.

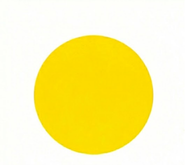
Stability phenomenon for perfect pancakes

Making a perfect pancake is hard, so judges allow for a margin of error:

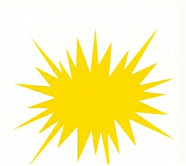
$$\varepsilon := P^2 - 4\pi A.$$

Question

If the margin of error ε is small, is the pancake "close" (in some suitable sense) to a circle?



Perfect pancake



Not perfect pancake

Bonnesen's inequality

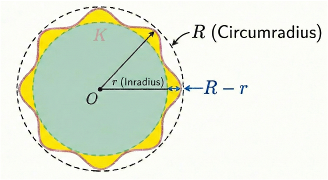
Theorem (Bonnesen (1921))

For a closed 2-D region,

$$(P^2 - 4\pi A) \geq \pi^2 (R - r)^2,$$

where r is inradius and R is circumradius.

If the margin of error is small, the pancake fits snugly between two circles that are close together.



Three stages of understanding an inequality

Stage 1: Established that inequality holds.

Stage 2: Characterized the equality cases.

Stage 3: Near-equality imply quantitative
proximity to the equality cases.

Stage 3 is known as the **stability phenomenon**.

Success Story:
Brunn–Minkowski–Lyusternik Inequality

Brunn–Minkowski–Lyusternik (BML) inequality

Theorem

For measurable $A, B \subset \mathbb{R}^n$,

$$\mathrm{Vol}(A+B)^{1/n} \;\geq\; \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n},$$

where $A+B$ *is Minkowski sum and* Vol *is Lebesgue measure.*

- Brunn (1887): convex bodies, $n \leq 3$;
- Minkowski (1896): convex bodies, all n's;
- Lyusternik (1935): non-convex bodies.

BML inequality, equivalent statement

Theorem

For measurable $A, B \subset \mathbb{R}^n$ *with unit volume,*

$$\mathrm{Vol}(tA + (1-t)B) \;\geq\; 1,$$

where $t \in (0, 1/2]$.

This formulation allows for a cleaner statement of the equality cases.

Equality conditions of BML inequality

Theorem

Equality $\mathrm{Vol}(tA + (1-t)B) = 1$ *holds,*

if and only if

up to translation, there is convex $K \supset A \cup B$ *so that*

$$\mathrm{Vol}(K \setminus A) \ = \ \mathrm{Vol}(K \setminus B) \ = \ 0.$$

Proved for convex case by Minkowski (1896), and for general case by Henstock–Macbeath (1953).

Stability of BML inequality

Theorem (Figalli–van Hintum–Tiba (2023))

Let $\delta := \mathrm{Vol}(tA + (1-t)B) \; - \; 1.$

If δ *is sufficiently small, then up to translation,*

there exists a convex $K \supset A \cup B$ *so that,*

$$\mathrm{Vol}(K \setminus A) + \mathrm{Vol}(K \setminus B) \quad \leq \quad C\sqrt{\delta / t},$$

for some $C := C_n > 0$.

Following a long series of works, this conclusively resolves the stability question for BLM inequality.

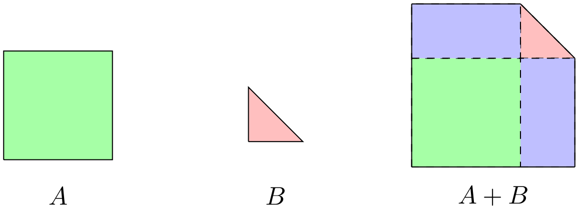
Unresolved Story:
Alexandrov–Fenchel Inequality

Mixed volumes: dimension 2

For convex $A, B \subset \mathbb{R}^2$,

$$\mathrm{Vol}(aA + bB) = V(A, A)\,a^2 + V(B, B)\,b^2 + 2V(A, B)\,ab$$

is a quadratic polynomial in $a, b \geq 0$.



Coefficients $V(K, K)$, $V(L, L)$, $V(K, L)$

are mixed volumes.

Mixed volumes: general dimensions

Theorem (Minkowski 1903)

For convex $C_1, \ldots, C_n \subset \mathbb{R}^n$, the function

$$(\lambda_1, \ldots \lambda_n) \mapsto \mathrm{Vol}(\lambda_1 C_1 + \ldots + \lambda_n C_n)$$

is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n \geq 0$.

Mixed volume $V(C_1, \ldots, C_n)$ is $\frac{1}{n!}$ of the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial expansion of $\mathrm{Vol}(\lambda_1 C_1 + \ldots + \lambda_n C_n)$.

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Equivalently, $V(C_1, \ldots, C_n)$ is $\frac{1}{n!}$ of intersection number of corresponding nef divisors $D_1, \ldots, D_n$.

Alexandrov-Fenchel (AF) inequality

Theorem

For convex $A, B, C_1, \ldots, C_{n-2} \subset \mathbb{R}^n$,

$$V^*(A,B)^2 \;\geq\; V^*(A,A)\, V^*(B,B),$$

where $V^*(A,B) \;:=\; V(A,B,C_1,\ldots,C_{n-2})\,.$

- Fenchel (1936): Announced the result, no rigorous proof.
- Alexandrov (1937): Provided rigorous proof(s).

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Modern perspective: AF inequality can be viewed as a consequence of volume polynomials being Lorentzian (Brändén–Huh (2020)).

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Note: Alexandrov–Fenchel inequality is false without assuming convexity.

Note: Brunn–Minkowski (convex) is consequence of Alexandrov–Fenchel inequality.

Equality conditions of AF inequality

Question (Alexandrov 1937)

Classify $A, B, C_1, \ldots, C_{n-2} \subset \mathbb{R}^n$ *for which*

$$V^*(A,B)^2 = V^*(A,A)\,V^*(B,B).$$

Bad news: Still open for general convex bodies;

Good news: Recently solved for convex polytopes.

Equality condition for convex polytopes

Theorem (Shenfeld-van Handel 2023)

Let $A, B, C_1, \ldots, C_{n-2}$ *be convex polytopes. Then*

$$V^*(A,B)^2 = V^*(A,A)\,V^*(B,B)$$

arises from a combination of three mechanisms:

- *Translation and scaling;*
- *Relative positions of normal cones of boundaries of* $C_1, \ldots, C_{n-2}$;
- *Relative positions of affine hulls of* $C_1, \ldots, C_{n-2}$.

Complexity of AF equality

Consider the decision problem:

Input: unimodular polytopes $A, B, C_1, \ldots, C_{n-2}$;

Output: - YES if $V^*(A,B)^2 = V^*(A,A)\,V^*(B,B)$;

- NO if $V^*(A,B)^2 > V^*(A,A)\,V^*(B,B)$.

Theorem (C.–Pak 2024)

This decision problem cannot be solved in polynomial time, unless NP $=$ coNP *.*

Key takeaway: AF equality condition is extremely complicated.

Stability of Alexandrov–Fenchel inequality

Quote (Shenfeld–van Handel 2024)

"It may be difficult to imagine what [stability of AF inequality for convex polytopes] could even look like given the large number of equality cases."

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To identify the source of difficulty, we analyze the stability of a special case of AF inequality.

Quadratic Permanent Inequality

Quadratic permanent (QP) inequality

The permanent of $n \times n$ matrix $M := (m_{i,j})$ is

$$\operatorname{per}(M) \;=\; \sum_{\sigma \in S_n} m_{1,\sigma(1)}\, m_{2,\sigma(2)} \cdots m_{n,\sigma(n)}.$$

Theorem (Egoryčev 1980, Falikman 1981)

For nonnegative $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $V \in \mathbb{R}^{n \times (n-2)}$,

$$\operatorname{per}(\mathbf{a}, \mathbf{b}, V)^2 \;\geq\; \operatorname{per}(\mathbf{a}, \mathbf{a}, V)\, \operatorname{per}(\mathbf{b}, \mathbf{b}, V).$$

Proof: Apply AF inequality to n-rectangles.

Motivation: Van der Waerden's conjecture

Conjecture (Van der Waerden 1926)

For nonnegative $n \times n$ matrix M, with row and column sums equal to 1,

$$\mathrm{per}(M) \;\geq\; \frac{n!}{n^n}.$$

Equality occurs $\iff$ *all entries of M equal to $1/n$.*

Falikman (1981) proved the inequality, Egoryčev (1980) proved the inequality and equality conditions.

Both proofs are based on QP inequality.

Equality conditions of QP inequality

Theorem (Panov 1984)

Suppose $\mathrm{per}(\mathbf{a}, \mathbf{a}, V), \mathrm{per}(\mathbf{a}, \mathbf{b}, V) > 0$. *Then*

$$\mathrm{per}(\mathbf{a}, \mathbf{b}, V)^2 = \mathrm{per}(\mathbf{a}, \mathbf{a}, V)\,\mathrm{per}(\mathbf{b}, \mathbf{b}, V)$$

if and only if

- *Every maximal zero submatrix* $V_{J,K}$ *of* V *satisfies* $|J| + |K| = n - 1$;
- $\mathrm{per}(\mathbf{a}_{J^c}, V_{J^c,K}) = \lambda \cdot \mathrm{per}(\mathbf{b}_{J^c}, V_{J^c,K})$ *for every such submatrix, where* $\lambda := \frac{\mathrm{per}(\mathbf{a},\mathbf{b},V)}{\mathrm{per}(\mathbf{b},\mathbf{b},V)}$; *and*
- $\mathbf{a}_j = \lambda \cdot \mathbf{b}_j$ *for all* j*'s in the intersection of these* J*'s.*

Equality conditions of QP inequality (informal)

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permanents of submatrices of $\mathbf{a}, \mathbf{b}, V$ *satisfy linear relations that are determined by the maximal zero submatrices of* V.

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Modern perspective: These conditions are manifestations of mechanism (1) and (3) of equality conditions of AF inequality.

Stability of QP inequality

Theorem (C.–Pak 2026+)

Suppose $\mathrm{per}(\mathbf{a}, \mathbf{a}, V), \mathrm{per}(\mathbf{b}, \mathbf{b}, V) > 0$ *and*

$$\delta \; := \; \frac{\mathrm{per}(\mathbf{a}, \mathbf{b}, V)}{\sqrt{\mathrm{per}(\mathbf{a}, \mathbf{a}, V)\,\mathrm{per}(\mathbf{b}, \mathbf{b}, V)}} - 1.$$

Then there exists $C := C_{n,V} > 0$ *so that*

$$\left| \frac{\mathbf{a}}{|\,\mathbf{a}\,|} - \mathbf{a}^{\circ} \right| + \left| \frac{\mathbf{b}}{|\,\mathbf{b}\,|} - \mathbf{b}^{\circ} \right| \; \leq \; C\sqrt{\delta},$$

where $\mathbf{a}^{\circ}$, $\mathbf{b}^{\circ}$ *are solutions to*

$$\mathrm{per}(\mathbf{a}^{\circ}, \mathbf{b}^{\circ}, V)^2 = \mathrm{per}(\mathbf{a}^{\circ}, \mathbf{a}^{\circ}, V)\,\mathrm{per}(\mathbf{b}^{\circ}, \mathbf{b}^{\circ}, V).$$

Stability of QP inequality

Proof involved analyzing mechanisms (1) and (3) in AF equality conditions, combined with spectral analysis unique to permanents.

Key takeaway: The difficulty in AF stability lies in quantifying the combined impact of mechanisms (1), (2), and (3).

Open problems

Mason's conjecture (strongest form)

Theorem

Let $\mathcal{M}$ be a matroid with n elements. For $k \geq 1$:

$$(\mathrm{I}_k)^2 \;\geq\; (1 + \tfrac{1}{k})(1 + \tfrac{1}{n-k})\,\mathrm{I}_{k+1}\,\mathrm{I}_{k-1},$$

where I_k is number of independent sets with k elements.

- Conjectured by Mason (1972);
- Proved independently by Brändén–Huh (2020) and Anari–Liu-Oveis Gharan–Vinzant (2024).

Equality conditions of Mason's conjecture

Theorem

Equality holds

$$(\mathrm{I}_k)^2 = \left(1+\tfrac{1}{k}\right)\left(1+\tfrac{1}{n-k}\right)\mathrm{I}_{k+1}\,\mathrm{I}_{k-1}$$

if and only if

$$\mathrm{I}_{k+1} \;=\; \binom{n}{k+1}.$$

- Proved by Murai–Nagaoka–Yazawa (2021);
- Later alternative proof by C.–Pak (2024).

Stability of Mason's conjecture

Question

Let

$$\delta \; := \; (\mathrm{I}_k)^2 \; - \; (1 + \tfrac{1}{k})(1 + \tfrac{1}{n-k})\, \mathrm{I}_{k+1}\, \mathrm{I}_{k-1}.$$

Is it true that

$$\frac{\mathrm{I}_{k+1}}{\binom{n}{k+1}} \; \geq \; 1 \; - \; f(\delta),$$

where $f(\delta) \to 0$ *as* $\delta \to 0$*?*

Progress on this question may pave the way to resolving a problem in complexity theory.

THANK YOU!

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Complexity of Mason's conjecture

Conjecture (Pak 2019)

$$\binom{n}{k+1}\binom{n}{k-1}\mathrm{I}_k^2 \; - \; \binom{n}{k}^2 \mathrm{I}_{k+1}\mathrm{I}_{k-1} \notin \#\mathsf{P}.$$

#P is the complexity class for counting solutions to NP problems.

The complexity proof for QP stability offers hope for resolving this conjecture via the stability of Mason's conjecture.

Complexity of stability of QP inequality

Consider the computational problem:

Input: $\mathbf{a}, \mathbf{b} \in \{0,1\}^n$, $V \in \{0,1\}^{n\times(n-2)}$;

Output: vectors $\mathbf{a}^\circ$, $\mathbf{b}^\circ$ satisfying

$$\left|\frac{\mathbf{a}}{|\,\mathbf{a}\,|} - \mathbf{a}^\circ\right| + \left|\frac{\mathbf{b}}{|\,\mathbf{b}\,|} - \mathbf{b}^\circ\right| \leq C\sqrt{\delta},$$

$$\mathrm{per}(\mathbf{a}^\circ, \mathbf{b}^\circ, V)^2 = \mathrm{per}(\mathbf{a}^\circ, \mathbf{a}^\circ, V)\,\mathrm{per}(\mathbf{b}^\circ, \mathbf{b}^\circ, V).$$

Theorem (C.–Pak 2026+)

There are no algorithms to find $\mathbf{a}^\circ, \mathbf{b}^\circ$ *that runs in poly(n) times, unless* NP = coNP *.*