Log-concavity for order-preserving permutations and maps

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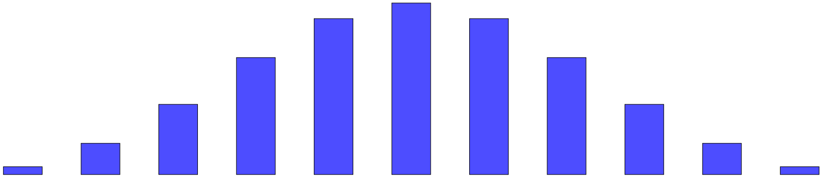
joint with Igor Pak and Greta Panova



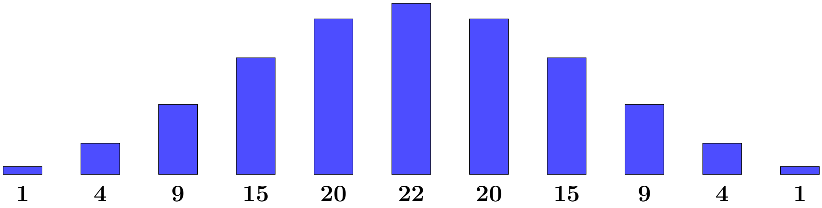
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad \text{for all } 1 < k < n.$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \;\leq\; \cdots \;\leq\; a_m \;\geq\; \cdots \;\geq\; a_n \quad \text{for some } 1 \leq m \leq n.$$



Example: binomial coefficients

$$a_k \;=\; \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\, a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

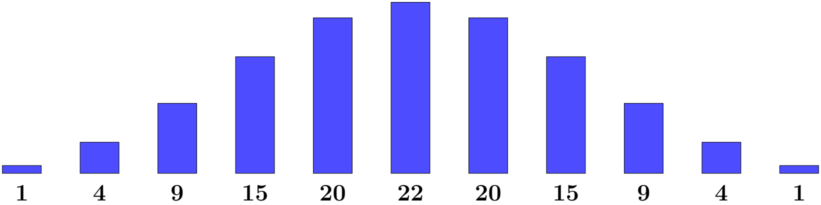
$$a_k \;=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions,}$$

where inversion of π is pair $x < y$ s.t. $\pi(x) > \pi(y)$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; (1+q)\,\ldots\,(1+q\ldots+q^{n-1})$$

is a product of log-concave polynomials.

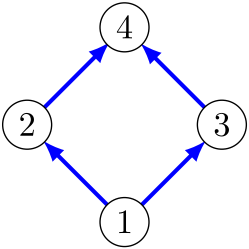


Log-concavity is a widespread phenomenon observed in numerous subjects in mathematics.

Today we focus on log-concavity for **permutations** arising from **posets**.

Partially ordered sets (posets)

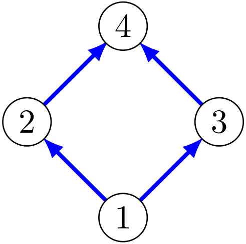
A (finite) poset P is a set $\{1, \ldots, n\}$
with a given partial order $\prec$ on the set.

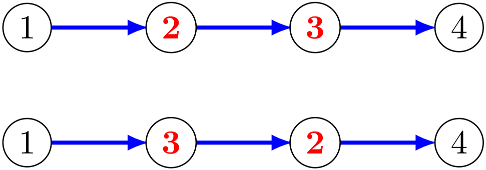


Oder-preserving permutations (linear extension)

A permutation $\pi : [n] \to [n]$ is order-preserving if

$$x \prec y \qquad \text{implies} \qquad \pi(x) \leq \pi(y).$$





Note that π can be viewed as completion of $\prec$.

Stanley inequality: simple form

Fix $x \in P$.

$$p_k \text{ is probability that } \mathcal{L}(x) = k,$$

where $\mathcal{L}$ is uniform random linear extension of P.

Theorem (Stanley '81)

For $k \geq 1$,

$$p_k{}^2 \;\geq\; p_{k+1}\, p_{k-1}.$$

The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Stanley inequality: generalized form

Fix $d \geq 0$, $x, y_1, \ldots, y_d \in P$ and $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

$p_k^{(d)}$ *is* probability $\mathcal{L}(x) = k$, $\mathcal{L}(y_i) = \ell_i$ for $i \in [d]$,

where $\mathcal{L}$ is uniform random linear extension of P.

Theorem (Stanley '81)

For $k \geq 1$,

$$\left(p_k^{(d)}\right)^2 \;\geq\; p_{k+1}^{(d)}\, p_{k-1}^{(d)}.$$

This inequality plays a vital role in the discovery of best known bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture.

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For incomplete partial order, there exist $x, y \in P$:

$$\frac{1}{3} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{2}{3},$$

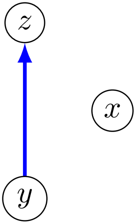
where $\mathcal{L}$ is uniform random linear extension of P.

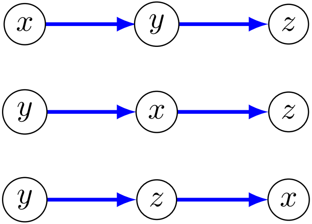
Quote (Brightwell-Felsner-Trotter '95)

"This problem remains one of the most intriguing problems in the combinatorial theory of posets."

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper,lower bound are achieved by this poset:





$$\mathbb{P}\left[\mathcal{L}(x) < \mathcal{L}(y)\right] = \frac{1}{3}; \qquad \mathbb{P}\left[\mathcal{L}(y) < \mathcal{L}(x)\right] = \frac{2}{3}.$$

Earliest known bound

Theorem (Kahn-Saks '84)

For incomplete partial order, there exist $x, y \in P$:

$$\frac{3}{11} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{8}{11},$$

roughly between 0.273 *and* 0.727.

Proof used log-concavity as a crucial component.

Best known bound

Theorem (Brightwell-Felsner-Trotter '95)

For incomplete partial order, there exist $x, y \in P$:

$$\frac{5-\sqrt{5}}{10} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{5+\sqrt{5}}{10},$$

roughly between 0.276 *and* 0.724.

This bound cannot be improved for infinite posets.

Log-concavity also plays crucial component in this proof.

Log-concavity comes from black box

In every proof of bounds for $\frac{1}{3}-\frac{2}{3}$ Conjecture, log-concavity played crucial yet **mysterious** roles.

This raises the question if there is a **less mysterious** explanation for this log-concavity.

Graham conjecture

Quote (Graham '83)

"[*Log-concavity of order-preserving permutations and order-preserving maps*] *should have a proof based on the* **FKG or AD inequalities**. *However, such a proof has up to now successfully eluded all attempts to find it*".

Kleitman/Harris/FKG/AD inequalities

Theorem (Kleitman '66)

For increasing subsets $A, B \subseteq 2^{[n]}$,

$$\frac{|A \cap B|}{2^n} \geq \frac{|A|}{2^n} \frac{|B|}{2^n}.$$

Example

For any $a, b, c, d \in V$ *in Erdös–Renyi random graph,*

$$\mathbb{P}\big[a \leftrightarrow b,\, c \leftrightarrow d\big] \geq \mathbb{P}\big[a \leftrightarrow b\big]\, \mathbb{P}\big[c \leftrightarrow d\big],$$

where $a \leftrightarrow b$ *is event that a and b are connected.*

Application of FKG inequality

Theorem (XYZ inequality, Shepp '82)

For incomparable elements $x, y, z \in P$:

$$\mathbb{P}[x \prec z \mid x \prec y] \quad \geq \quad \mathbb{P}[x \prec z].$$

Intuition

A baseball team losing this week increases the likelihood for the team to lose next week.

Application of FKG inequality

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Intuition

A baseball team losing this week increases the likelihood for the team to lose next week.

Recommended survey on the subject

Winkler '86: Correlation and Order, *Contemp. Math.*

Back to Graham conjecture

Quote (Graham '83)

"[*Log-concavity of order-preserving permutations and* **order-preserving maps**] *should have a proof based on the FKG or AD inequalities.*"

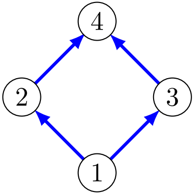
We will first focus on order-preserving maps.

Order-preserving maps

Fix a poset P.

A map $f : P \to \{1, \ldots, n\}$ is order-preserving if

$$x \prec y \qquad \text{implies} \qquad f(x) \leq f(y).$$



	1	2	3	4
f	1	2	1	2

✓

	1	2	3	4
f	2	1	3	4

✗

Linear extensions are order-preserving maps that are also bijections.

Log-concavity for order-preserving maps

Fix $x \in P$.

$$q_k \ \text{ is } \ \text{probability that } \mathcal{M}(x) = k,$$

where $\mathcal{M}$ is uniform random order-preserving map.

Conjecture (Graham '83)

For $k \geq 1$,

$$q_k^2 \ \geq \ q_{k+1}\, q_{k-1}.$$

This is the analogue of Stanley inequality for order-preserving maps.

Log-concavity for order-preserving maps

Theorem (Daykin–Daykin–Paterson '84)

For every $k \geq 1$,

$$q_k^2 \;\geq\; q_{k+1}\, q_{k-1}.$$

Proof used an explicit injective argument,
not based on FKG/AD inequality.

Quote (Daykin–Daykin–Paterson '84)

"[*Proof using FKG or Ahlswede–Daykin inequality*] *have as yet eluded discovery*".

Answer to second part of Graham conjecture

Theorem 1 (C.–Pak–Panova '23, C.–Pak '23)

New proof of Daykin–Daykin–Paterson inequality based on Ahlswede–Daykin inequality, which generalizes to multi-weighted version. □

This validates Graham's prediction for order-preserving maps.

Answer to second part of Graham conjecture

Theorem 1 (C.–Pak–Panova '23, C.–Pak '23)

New proof of Daykin–Daykin–Paterson inequality based on Ahlswede–Daykin inequality, which generalizes to multi-weighted version. □

This validates Graham's prediction for order-preserving maps.

Question

Can we do the same thing for **linear extensions***?*

Progress on first part of Graham conjecture

Theorem 2 (C.–Pak '23+)

Generalized Stanley inequality for linear extensions,

$$\left(p_k^{(d)}\right)^2 - p_{k+1}^{(d)} p_{k-1}^{(d)} \geq 0$$

does not belong *to complexity class* **#P** *if* $d \geq 2$*,*
unless polynomial hierarchy collapses.

On the other hand, it is known that Kleitman inequality **belongs** to complexity class **#P**.

Conclusion

Quote (Graham '83)

"[*Log-concavity of order-preserving permutations and order-preserving maps*] *should have a proof based on the FKG or AD inequalities.*

- YES for order-preserving maps.
- Unknown for order-preserving permutations, but Kleitman inequality is probably not enough.

THANK YOU!

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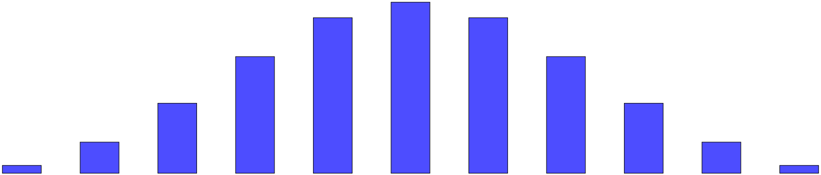

FKG or FGK?

Commun. math. Phys. 22, 89–103 (1971)

Correlation Inequalities on Some Partially Ordered Sets

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Stanley (poset) inequality: consequence

Weak Bruhat order on permutation group S_n is

$\pi \trianglelefteq \sigma$ if some reduced word of π is a left subword of some reduced word of σ.

For $\sigma \in S_n$, let

$N^\sigma(k)$:= number of $\pi \in S_n$ such that $\pi \trianglelefteq \sigma$ and $\pi(1) = k$.

Corollary

Sequence $N^\sigma(1), \ldots, N^\sigma(n)$ is log-concave.

Making changes to $\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For incomplete partial order, there exist $x, y \in P$:

$$\frac{1}{3} \quad \leq \quad \mathbb{P}\big[\,\mathcal{L}(x) < \mathcal{L}(y)\,\big] \quad \leq \quad \frac{2}{3},$$

where $\mathcal{L}$ is uniform random linear extension of P.

$\frac{1}{3}$–$\frac{2}{3}$ bound might be relevant only to "small" posets.

For "large" posets both sides should converge to $\frac{1}{2}$.

Kahn-Saks Conjecture

$\delta(P)$ is largest number so that there are $x, y \in P$:

$$\delta(P) \ \leq \ \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \ \leq \ 1 - \delta(P).$$

$\frac{1}{3} - \frac{2}{3}$ Conjecture is equivalent to
$\delta(P) \geq \frac{1}{3}$ for P not completely ordered.

Kahn-Saks Conjecture

Conjecture (Kahn-Saks '84)

$$\delta(P) \to \frac{1}{2} \quad \textit{as} \quad \text{width}(P) \to \infty.$$

width(P) is maximum cardinality of a subset of incomparable elements.

Komlós '90 proved Conjecture for posets with $\Omega(\frac{n}{\log\log\log n})$ minimal elements.

C.-Pak-Panova '21 proved Conjecture for Young diagram posets with fixed width.

Ahlswede–Daykin inequality

L is a finite distributive lattice.

$f_1, f_2, f_3, f_4 : L \to \mathbb{R}_{\geq 0}$ are nonnegative functions.

Theorem (Ahlswede–Daykin '78)

Suppose that

$$f_1(x)\, f_2(y) \;\leq\; f_3(x \vee y)\, f_4(x \wedge y) \qquad \forall x, y \in L.$$

Then

$$f_1(L)\, f_2(L) \;\leq\; f_3(L)\, f_4(L).$$

Proof of Daykin–Daykin–Paterson inequality

Let L be the distributive lattice consisting of order-preserving functions $g : P \to \{0, 1, \ldots, n\}$.

The join and meet operation are

$$(g_1 \wedge g_2)(z) := \max\{g_1(z), g_2(z)\},$$
$$(g_1 \vee g_2)(z) := \min\{g_1(z), g_2(z)\},$$

for $g_1, g_2 \in L$ and $z \in P$.

Proof of Daykin–Daykin–Paterson inequality

The four functions $f_1, f_2, f_3, f_4 : L \to \mathbb{R}_{\geq 0}$ are

$$
\begin{aligned}
f_1(g) &:= \mathbb{1}\{g(x) = k-1 \text{ and } g(z) \geq 1 \ \forall z \in P\},\\
f_2(g) &:= \mathbb{1}\{g(x) = k \text{ and } g(z) \leq n-1 \ \forall z \in P\},\\
f_3(g) &:= \mathbb{1}\{g(x) = k \text{ and } g(z) \geq 1 \ \forall z \in P\},\\
f_4(g) &:= \mathbb{1}\{g(x) = k-1 \text{ and } g(z) \leq n-1 \ \forall z \in P\}
\end{aligned}
$$

By using translation invariance,

$$
\begin{aligned}
f_1(L) &= q_{k-1}, & f_2(L) &= q_{k+1},\\
f_3(L) &= q_k, & f_4(L) &= q_k.
\end{aligned}
$$

Conclusion now follows from AD inequality. □