Complexity of Combinatorial Log-concave Inequalities

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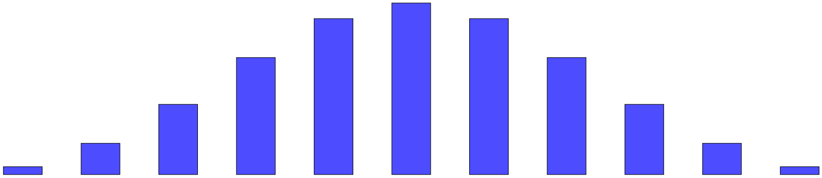
joint with Igor Pak



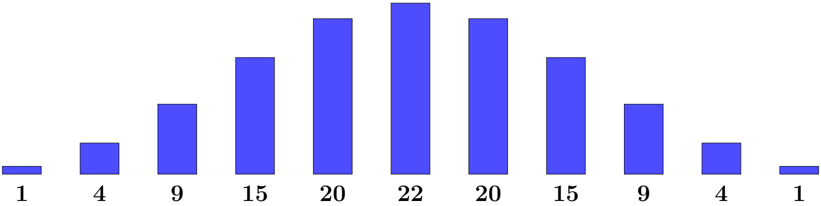
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \geq a_{k+1}\, a_{k-1} \qquad (1 < k < n).$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \leq \cdots \leq a_m \geq \cdots \geq a_n \quad \text{for some } 1 \leq m \leq n.$$



Example: binomial coefficients

$$a_k \;=\; \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\,a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

$$a_k \;=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; \prod_{i=1}^{n-1} (1 + q + q^2 + \ldots + q^i)$$

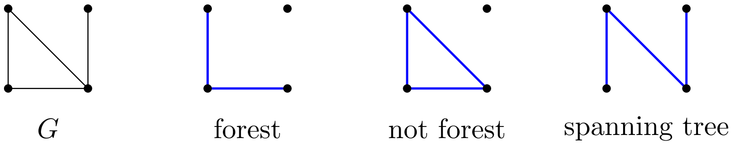
is a product of log-concave polynomials.

Examples: forests of a graph

a_k = number of forests with k edges of graph G.

Forest is a subset of edges of G that has no cycles.

Log-concavity was conjectured for all matroids (Mason '72), and was proved through combinatorial Hodge theory (Huh '15).



Log-concavity has been proved in various areas of mathematics in varying degrees of “complexities”.

We would like to rigorously formalize this **difference in complexities**.

Log-concavity has been proved in various areas of mathematics in varying degrees of "complexities".

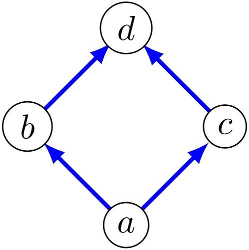
We would like to rigorously formalize this **difference in complexities**.

We will start with log-concave **poset inequalities**.

Poset inequalities

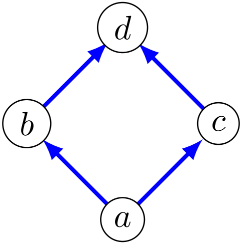
Partially ordered sets

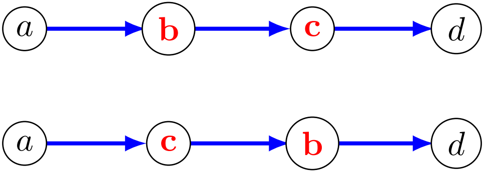
A poset $\mathcal{P}$ is a set X with a partial order $\prec$ on X.



Linear extension

A linear extension L is a complete order of $\prec$.





We write $L(x) = k$ if x is k-th smallest in L.

Stanley (poset) inequality: simple form

Fix $x \in \mathcal{P}$.

$N(k) :=$ number of linear extensions with $L(x) = k$.

Theorem (Stanley '81)

$$N(k)^2 \geq N(k+1)\,N(k-1) \qquad (k \in \mathbb{N}).$$

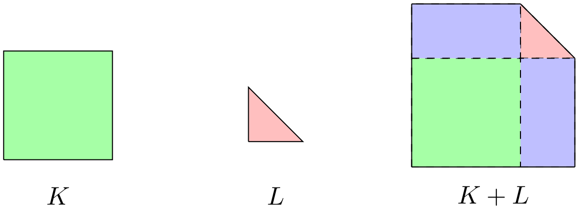
The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Mixed volumes: dimension 2

For convex bodies $K, L \subseteq \mathbb{R}^2$,

$$\mathrm{Vol}(aK+bL) = V(K,K)\,a^2 + V(L,L)\,b^2 + 2V(K,L)\,ab$$

is a quadratic polynomial in $a, b \geq 0$.



Coefficients $V(K,K)$, $V(L,L)$, $V(K,L)$

are mixed volumes.

Mixed volumes: dimension n

Theorem (Minkowski '03)

For convex bodies $K_1, \ldots, K_n \subseteq \mathbb{R}^n$, the function

$$(\lambda_1, \ldots \lambda_n) \mapsto \mathrm{Vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n)$$

is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n \geq 0$.

Mixed volume $V(K_1, \ldots, K_n)$ is $\frac{1}{n!}$ of the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial expansion of $\mathrm{Vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n)$.

Alexandrov-Fenchel (AF) inequality

Theorem (Alexandrov '37, Fenchel '36)

For convex bodies $A, B, K_1, \ldots, K_{n-2} \subseteq \mathbb{R}^n$,

$$V^*(A,B)^2 \;\geq\; V^*(A,A)\, V^*(B,B),$$

where $V^*(A,B) \;:=\; V\big(A, B, K_1, \ldots, K_{n-2}\big)\,.$

Stanley inequality $N(k)^2 \geq N(k+1)\, N(k-1)$
follows by substituting $A, B, K_1, \ldots, K_{n-2}$ with
slices of order polytopes.

Stanley (poset) inequality: true form

Fix $d \geq 0$, $x, y_1, \ldots, y_d \in \mathcal{P}$ and $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

$N_d(k) :=$ number of linear extensions with $L(x) = k, \quad L(y_i) = \ell_i \quad$ for $i \in [d]$.

Theorem (Stanley '81)

$$N_d(k)^2 \geq N_d(k+1)\, N_d(k-1) \qquad (k \in \mathbb{N}).$$

This form corresponds to imposing boundary conditions in PDE/statistical physics.

When is equality achieved?

Question (Stanley '81)

Find equality condition for [Stanley inequality].

Quote (Gardner '02)

If inequalities are silver currency in mathematics, those that come along with precise equality conditions are gold.

Equality condition: $d = 0$ (numerical)

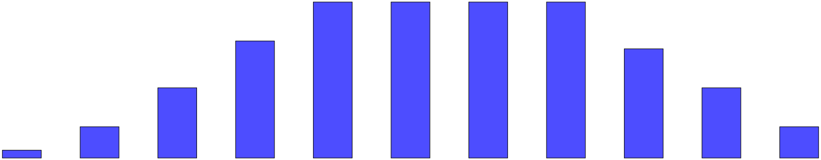
Theorem (Shenfeld-van Handel '23)

Suppose $d = 0$ *and* $N_d(k) > 0$. *Then*

$$N_d(k)^2 = N_d(k+1)\, N_d(k-1)$$

if and only if

$$N_d(k) \;=\; N_d(k+1) \;=\; N_d(k-1).$$



Equality condition: $d = 0$ (combinatorial)

Theorem (Shenfeld-van Handel '23)

Suppose $d = 0$ *and* $N_d(k) > 0$. *Then*

$$N_d(k)^2 = N_d(k+1)\, N_d(k-1)$$

if and only if

$$|\mathcal{P}_{<z}| > k \qquad \text{for all } z \in \mathcal{P}_{>x},$$

$$|\mathcal{P}_{>z}| > |P| - k + 1 \qquad \text{for all } z \in \mathcal{P}_{<x},$$

where $\mathcal{P}_{<z} :=$ *set of* $y \in \mathcal{P}$ *with* $y < z$.

This is a combinatorial condition, and can be checked in $O(|\mathcal{P}|^2)$ steps.

Equality condition: $d \geq 1$ (numerical)

Theorem (Ma–Shenfeld '24)

Suppose $d \geq 1$ *and* $N_d(k) > 0$. *Then*

$$N_d(k)^2 = N_d(k+1)\, N_d(k-1)$$

if and only if

$$N_d(k) \;=\; N_d(k+1) \;=\; N_d(k-1).$$

However, combinatorial equality condition was not extended to $d \geq 1$.

Main result

Consider the decision problem for checking equality in Stanley inequality:

$$N_d(k)^2 \overset{?}{=} N_d(k+1)\, N_d(k-1).$$

Theorem 1 (C.–Pak '23+)

- $d \leq 1$: *combinatorial equality condition that is checkable in* poly$(|\mathcal{P}|)$ *steps.*
- $d \geq 2$: *not part of* **polynomial hierarchy**, *unless polynomial hierarchy collapses.*

Polynomial hierarchy

Decision vs counting

Decision problem: answer is either '`Yes`' or '`No`'.

Counting problem: answer is a nonnegative integer.

Example (3-colorings of graph G)

- *Decision problem: Check if there exists a proper 3-coloring of G.*
- *Counting problem: Find the number of proper 3-colorings of G.*

Polynomial hierarchy is a subclass of decision problems.

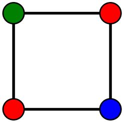
Complexity class P

$$P := \left\{ \begin{array}{l} \text{Decision problems solvable by deterministic} \\ \text{Turing machine in polynomial time} \end{array} \right\}$$

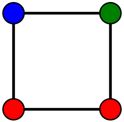
Example

Check if a given 3-coloring of a graph G is proper.

This can be solved in $O(n^2)$ time by checking the color of endpoints of every edge.



YES

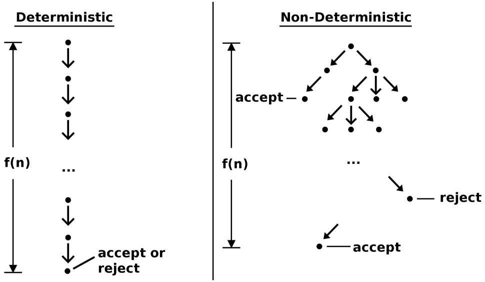


NO

Complexity class NP

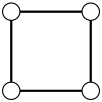
$$\mathrm{NP} := \left\{ \begin{array}{l} \text{Decision problems solvable by nondetermi-} \\ \text{nistic Turing machine in polynomial time} \end{array} \right\}.$$

- Can split into many parallel branches;
- Output 'YES' if one of the branches said 'YES';
- Output 'NO' if all branches said 'NO'.

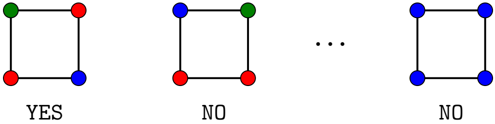


Complexity class NP: example

Problem: Check if graph G has a proper 3-coloring.



Each branch corresponds to checking if
a particular 3-coloring of G is proper.

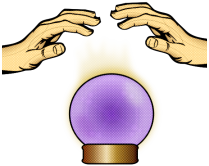


Output to this example is ‘YES’.

Turing machine with an oracle

At each step, this machine can either:

- Perform usual nondeterministic Turing machine operation; or
- Ask an oracle that is able to answer any instance of a given computational problem.



Turing machine with an oracle: example

Problem: Check if there is an induced subgraph of G of size $\lceil n/2 \rceil$ that is not 3-colorable.

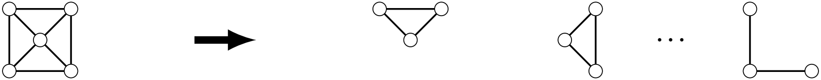
Oracle: Can check if a graph is 3-colorable.

Turing machine with an oracle: example

Problem: Check if there is an induced subgraph of G of size $\lceil n/2 \rceil$ that is not 3-colorable.

Oracle: Can check if a graph is 3-colorable.

Each branch of the machine corresponds to an induced subgraph of G of size $\lceil n/2 \rceil$.



For every branch, oracle checks if subgraph is 3-colorable.

Complexity class Σ_i^{P}

The first two classes are

$$\Sigma_0^{\mathsf{P}} \ := \ \mathsf{P}; \qquad \Sigma_1^{\mathsf{P}} \ := \ \mathsf{NP}.$$

For $i \geq 1$, the class $\Sigma_i^{\mathsf{P}} := \mathsf{NP}^{\Sigma_{i-1}^{\mathsf{P}}}$ is

$$\left\{ \begin{array}{l} \text{Decision problems solvable by nondetermi-} \\ \text{nistic Turing machine in polynomial time} \\ \text{with an oracle for problem from } \Sigma_{i-1}^{\mathsf{P}}. \end{array} \right\}.$$

Note that

$$\Sigma_0^{\mathsf{P}} \ \subseteq \ \Sigma_1^{\mathsf{P}} \ \subseteq \ \Sigma_2^{\mathsf{P}} \ \subseteq \ \Sigma_3^{\mathsf{P}} \ \subseteq \ \cdots$$

Complexity class Σ_i^{P}: example

Problem A: Check if a 3-coloring of G is proper.
Problem A is in $\Sigma_0^{\mathsf{P}} = \mathsf{P}$.

Problem B: Check if G has a proper 3-coloring.
Problem B is in $\Sigma_1^{\mathsf{P}} = \mathsf{NP}$.

Problem C: Check if there is an induced subgraph of G of size $\lceil n/2 \rceil$ that is not 3-colorable.
Problem C is in $\Sigma_2^{\mathsf{P}} = \mathsf{NP}^{\mathsf{NP}}$.

Polynomial hierarchy (PH)

Polynomial hierarchy is the union of all Σ_i^{P}'s,

$$\mathsf{PH} := \bigcup_{i=0}^{\infty} \Sigma_i^{\mathsf{P}}.$$

Conjecture

Polynomial hierarchy does not collapse,

$$\Sigma_0^{\mathsf{P}} \subsetneq \Sigma_1^{\mathsf{P}} \subsetneq \Sigma_2^{\mathsf{P}} \subsetneq \Sigma_3^{\mathsf{P}} \subsetneq \cdots$$

- $\Sigma_0^{\mathsf{P}} = \Sigma_1^{\mathsf{P}}$ is equivalent to $\mathsf{P} = \mathsf{NP}$.
- $\Sigma_1^{\mathsf{P}} = \Sigma_2^{\mathsf{P}}$ is equivalent to $\mathsf{NP} = \mathsf{coNP}$.

Back to main result

Consider the decision problem for checking equality in Stanley inequality:

$$N_d(k)^2 \overset{?}{=} N_d(k+1)\, N_d(k-1).$$

Theorem (C.–Pak '23+)

- $d \leq 1$: *Problem is in* P.
- $d \geq 2$: *Problem is not in* **PH**, *unless* PH *collapses.*

Alexandrov–Fenchel equality condition

Consider the decision problem for checking equality in AF inequality for unimodular polytopes:

$$V^*(A,B)^2 \stackrel{?}{=} V^*(A,A)\,V^*(B,B).$$

Theorem 2 (C.–Pak ‘23+)

Problem is not in **PH**, *unless* PH *collapses.*

Shenfeld–van Handel (‘23) obtained complete geometric description of AF equality, but those conditions are computationally intractable.

Recall our goal ...

Log-concavity has been proved in various aspects of mathematics in varying degrees of complexities.

We would like to rigorously formalize this **difference in complexities**.

Complexity class $\#P$

Complexity class $\#P$

$$\#\mathrm{P} := \left\{ \text{Counting problems realizable as number of `YES' branches in some nondeterministic Turing machine.} \right\}.$$

Example

Count number of proper 3-colorings of graph G.

Example

Count number of linear extensions $N_d(k)$ of poset $\mathcal{P}$.

Main result

Theorem 3 (C.–Pak '23+)
For $d \geq 2$, the defect of Stanley inequality

$$N_d(k)^2 \; - \; N_d(k+1)\, N_d(k-1)$$

is not in $\#P$, unless PH *collapses.*

Note: $N_d(k)^2$ and $N_d(k+1)\, N_d(k-1)$ are in $\#P$.

Theorem is a consequence of previous main result.

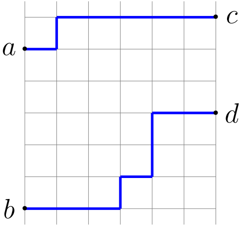
Example: defect of binomial inequalities

$$\binom{n}{k}^2 \geq \binom{n}{k+1}\binom{n}{k-1} \qquad (1 < k < n).$$

This inequality has a lattice path interpretation:

$K(a \to c, b \to d) :=$ no. of pairs of north-east lattice paths from a to c and b to d,

for $a, b, c, d \in \mathbb{Z}^2$.



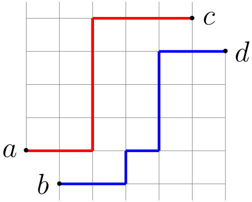
Example: defect of binomial inequalities

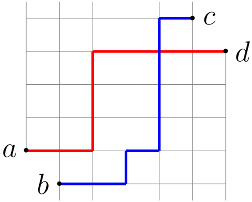
Let

$$a = (0,1), \qquad c = (k, n-k+1),$$
$$b = (1,0), \qquad d = (k+1, n-k).$$

Then

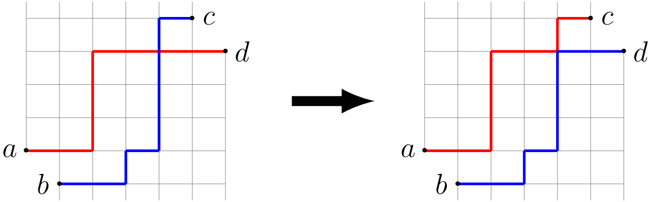
$$K(a \to c, b \to d) = \binom{n}{k}^2,$$
$$K(a \to d, b \to c) = \binom{n}{k-1}\binom{n}{k+1}.$$





Example: defect of binomial inequalities

Note $K(a \to c, b \to d) \geq K(a \to d, b \to c)$ by path-swapping injections.



$K(a \to c, b \to d) - K(a \to d, b \to c)$ is number of pairs of north-east lattice paths from a to c, b to d, that do not intersect.

This number is thus in $\#P$.

Example: Edge correlation for spanning trees

Let G be a graph, let e, f be distinct edges of G.

$$\begin{aligned} \mathcal{T} &:= \text{no. of spanning trees of } G, \\ \mathcal{T}_e &:= \text{no. of spanning trees of } G \text{ containing } e, \\ \mathcal{T}_{e,f} &:= \text{no. of spanning trees of } G \text{ containing } e \text{ and } f. \end{aligned}$$

Theorem

$$\mathcal{T}_e \mathcal{T}_f \;\geq\; \mathcal{T} \mathcal{T}_{e,f}.$$

Defect can be computed in polynomial time by matrix tree theorem. This number is thus in $\#P$.

Back to main result

Theorem (C.–Pak '23+)
For $d \geq 2$, the defect of Stanley inequality

$$N_d(k)^2 \ - \ N_d(k+1)\, N_d(k-1)$$

is not in $\#P$, unless PH *collapses.*

This differentiates Stanley inequality from binomial inequality and edge correlation inequality.

In particular, a combinatorial interpretation for defect of Stanley inequality is unlikely to exist.

Matroids

Object: matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphical matroids

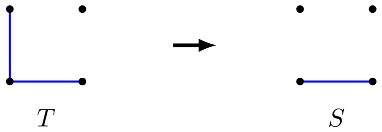
- X = edges of a graph G,
- $\mathcal{I}$ = forests in G.

Realizable matroids

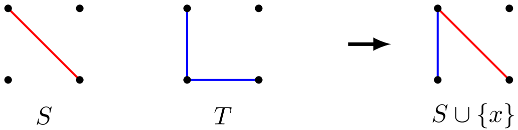
- X = finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I}$ = sets of linearly independent vectors.

Matroids: conditions

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.



- If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



A basis is a maximal independent set.

Rank r of matroid is the size of the bases.

Stanley–Yan inequality

Fix $d \geq 0$, disjoint subsets $S, S_1, \ldots, S_d$ of X, and $\ell_1, \ldots, \ell_d \in \mathbb{N}$.

$$\mathrm{B}_d(k) := \begin{array}{l} \text{number of bases } B \text{ of } \mathcal{M} \text{ such that} \\ |B \cap S| = k, \ |B \cap S_i| = \ell_i \ \text{ for } \ i \in [d], \end{array}$$

divided by $\binom{r}{k, \ell_1, \ldots, \ell_d}$.

Theorem (Stanley '81, Yan '23)

$$\mathrm{B}_d(k)^2 \ \geq \ \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1) \qquad (k \in \mathbb{N}).$$

Proved for regular matroids by (Stanley '81), and for all matroids by (Yan '23).

Stanley–Yan implies Mason

$\mathrm{I}(k) \;:=\;$ no. of independents sets with k elements.

$\mathrm{I}(k)$ is no. of forest with k edges for graphic matroids.

Theorem (Mason inequality)

$$\mathrm{I}(k)^2 \;\geq\; \mathrm{I}(k+1)\,\mathrm{I}(k-1).$$

Proof.

$\mathcal{M}$ = given matroid with ground set X.

$\mathcal{M}'$ = direct sum of $\mathcal{M}$ with the free matroid.

Set $d = 0$ and $S = X$. Then

$$\mathrm{I}(k) \text{ for } \mathcal{M} \quad = \quad \mathrm{B}_d(k) \text{ for } \mathcal{M}'.$$

Main result

Theorem 4 (C.–Pak)
For $d \geq 1$, defect of Stanley–Yan inequality for binary matroids

$$\mathrm{B}_d(k)^2 \;-\; \mathrm{B}_d(k+1)\,\mathrm{B}_d(k-1)$$

is not in $\#P$, unless PH *collapses.*

This differentiates Stanley–Yan inequality from binomial inequality and edge correlation inequality.

What is next?

Conjecture

For $d = 0$, *defect of Stanley inequality*

$$N(k)^2 \; - \; N(k+1)\, N(k-1) \quad \notin \; \#P.$$

For $d = 0$, *defect of Stanley–Yan inequality*

$$\mathrm{B}(k)^2 \; - \; \mathrm{B}(k+1)\, \mathrm{B}(k-1) \quad \notin \; \#P.$$

For $d = 0$, *defect of Mason inequality*

$$\mathrm{I}(k)^2 \; - \; \mathrm{I}(k+1)\, \mathrm{I}(k-1) \quad \notin \; \#P.$$

THANK YOU!

Preprint: www.arxiv.org/abs/2309.05764

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