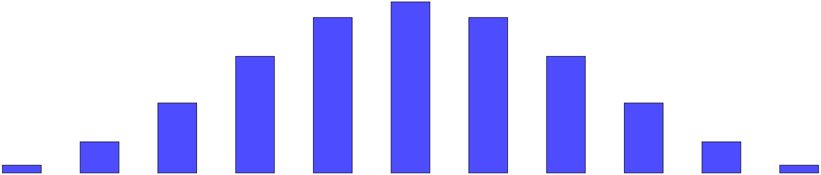
Log-concave Poset Inequalities

Day 3: Stanley's Poset Inequality

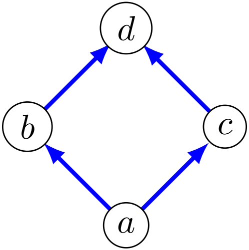
Swee Hong Chan

joint with Igor Pak



Partially ordered sets

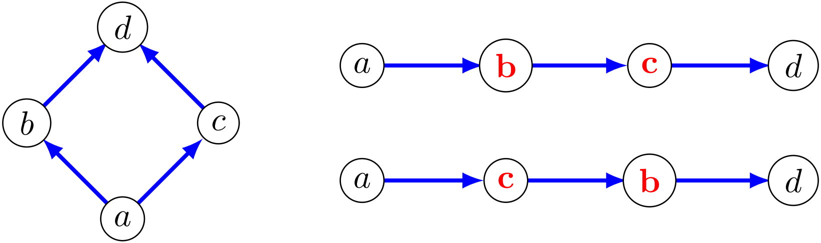
A poset $\mathcal{P}$ is a set X with a partial order $\prec$ on X.



We denote by $n := |X|$ the size of the poset.

Linear extension

A linear extension is a complete order of $\prec$.



Stanley's poset inequality (simple case)

Fix $z \in \mathcal{P}$. Let

$$N_k := \begin{array}{c}\text{number of linear extensions with} \\ z \text{ being } k\text{-th smallest.}\end{array}$$

Theorem (Stanley '81)

For every poset $\mathcal{P}$ and $k \in \{2, \ldots, n-1\}$,

$$N_k{}^2 \;\geq\; N_{k+1}\, N_{k-1}.$$

The inequality was originally proved using Aleksandrov-Fenchel inequality for mixed volumes, in a greater generality.

Stanley's poset inequality: a consequence

Weak Bruhat order on permutation group S_n is

$\pi \trianglelefteq \sigma$ if some reduced word of π is a left subword of some reduced word of σ.

For $\sigma \in S_n$, let

$$N_k^\sigma := \text{number of } \pi \in S_n \text{ such that } \pi \trianglelefteq \sigma \text{ and } \pi(1) = k\,.$$

Corollary

Sequence $N_1^\sigma, \ldots, N_n^\sigma$ *is log-concave.*

Stanley's poset inequality (simple case)

Fix $z \in \mathcal{P}$. Let

$$N_k \; := \; \begin{array}{c}\text{number of linear extensions with} \\ z \text{ being } k\text{-th smallest.}\end{array}$$

Theorem (Stanley '81)

For every poset $\mathcal{P}$ *and* $k \in \{2, \ldots, n-1\}$,

$$N_k{}^2 \; \geq \; N_{k+1} \, N_{k-1}.$$

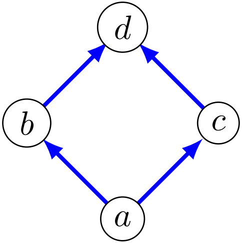
We will give a new proof using atlas method.

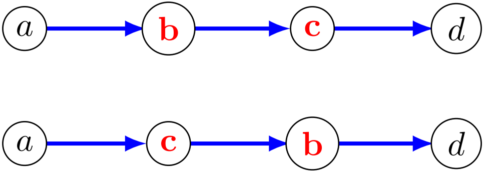
Extension matrix theorem

Linear extensions as words

Each linear extension corresponds to $\omega_1 \cdots \omega_n \in X^*$:

ω_i is i-th smallest element in the extension.





abcd and *acbd* are the two linear extensions.

Extension matrix: Definition part 1

Let $k \in \{2, \ldots, n-1\}$.

Let $x_1, \ldots, x_\ell$ be minimal elements of $\mathcal{P}$.

Let $x_{\ell+1}, \ldots, x_m$ be maximal elements of $\mathcal{P}$.

Extension matrix $\mathrm{E}[\mathcal{P}, k]$ is $m \times m$ matrix where, for $i \in [\boldsymbol{\ell}]$, $j \in [\boldsymbol{m}] \setminus [\boldsymbol{\ell}]$:

$$(\mathrm{E}[\mathcal{P}, k])_{i,j} \;:=\; \begin{array}{l}\text{no. of linear extensions } \omega \text{ with} \\ z = \omega_{\boldsymbol{k}} \text{ and } x_i = \omega_{\mathbf{1}} \text{ and } x_j = \omega_{\boldsymbol{n}}.\end{array}$$

Here i corresponds to a minimal element,

and j corresponds to a maximal element,

and every extension has z as $\boldsymbol{k}$-th smallest.

Extension matrix: Definition part 2

For distinct $i, j \in [\boldsymbol{\ell}]$:

$(\mathrm{E}[\mathcal{P}, k])_{i,j} :=$ no. of lin. extensions ω with $z = \omega_{\boldsymbol{k+1}}$ and $x_i = \omega_{\mathbf{1}}$ and $x_j = \omega_{\mathbf{2}}$;

$(\mathrm{E}[\mathcal{P}, k])_{i,i} :=$ no. of lin. extensions ω with $z = \omega_{\boldsymbol{k+1}}$ and $x_i = \omega_{\mathbf{1}}$ and $\omega_{\mathbf{1}} \prec \omega_{\mathbf{2}}$ in $\mathcal{P}$.

Here i, j corresponds to minimal elements of $\mathcal{P}$, and every extension has z as $(\boldsymbol{k} + \mathbf{1})$-th smallest.

Extension matrix: Definition part 3

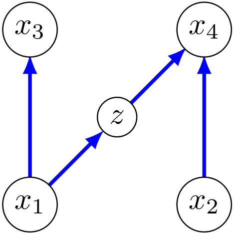
For distinct $i, j \in [\boldsymbol{m}] \setminus [\boldsymbol{\ell}]$:

$(\mathrm{E}[\mathcal{P}, k])_{i,j}$:= no. of lin. extensions ω with $z = \omega_{\boldsymbol{k-1}}$ and $x_i = \omega_{\boldsymbol{n}}$ and $x_j = \omega_{\boldsymbol{n-1}}$;

$(\mathrm{E}[\mathcal{P}, k])_{i,i}$:= no. of lin. extensions ω with $z = \omega_{\boldsymbol{k-1}}$ and $x_i = \omega_{\boldsymbol{n}}$ and $\omega_{\boldsymbol{n-1}} \prec \omega_{\boldsymbol{n}}$ in $\mathcal{P}$.

Here i, j corresponds to maximal elements of $\mathcal{P}$, and every extension has z as $(\boldsymbol{k - 1})$-th smallest.

Extension matrix: Example ($k = 3$)



$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

$\mathrm{E}[\mathcal{P}, 3]$

$$(\mathrm{E}[\mathcal{P}, 3])_{1,4} := |\{x_1 x_2 z x_3 x_4, x_1 x_3 z x_2 x_4\}| = 2;$$
$$(\mathrm{E}[\mathcal{P}, 3])_{1,2} := |\{x_1 x_2 x_3 z x_4, x_1 x_2 x_4 z x_3\}| = 2;$$
$$(\mathrm{E}[\mathcal{P}, 3])_{1,1} := |\{x_1 x_3 x_2 z x_4\}| = 1;$$
$$(\mathrm{E}[\mathcal{P}, 3])_{3,4} := |\{x_1 z x_2 x_3 x_4\}| = 1;$$
$$(\mathrm{E}[\mathcal{P}, 3])_{4,4} := |\{x_1 z x_3 x_2 x_4\}| = 1.$$

Recap: Hyperbolic property

M has hyperbolic property (Hyp) if

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, M\boldsymbol{x} \rangle \langle \boldsymbol{y}, M\boldsymbol{y} \rangle$$

for every $\boldsymbol{x} \in \mathbb{R}^r$ and $\boldsymbol{y} \in \mathbb{R}^r_{\geq 0}$.

M satisfies (OPE) if

$\boldsymbol{M}$ has at most one positive eigenvalue.

Lemma (Lemma 3.5 (C.–Pak 22))

M satisfies (Hyp) $\iff$ *M satisfies* (OPE).

Extension matrix theorem

Theorem

For every poset $\mathcal{P}$ *and* $k \in \{2, \ldots, n-1\}$,

The matrix $\mathrm{E}[\mathcal{P}, k]$ *satisfies* (Hyp).

This theorem implies Stanley's inequality.

Extension matrix thm implies Stanley's inequality

Let

$$M \;:=\; \mathrm{E}[\mathcal{P}, k] \quad \boldsymbol{x} \;:=\; (\mathbf{1}^{\ell}, \mathbf{0}^{m-\ell}), \quad \boldsymbol{y} \;:=\; (\mathbf{0}^{\ell}, \mathbf{1}^{m-\ell}).$$

Then

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle \;=\; N_k, \;\; \langle \boldsymbol{x}, M\boldsymbol{x} \rangle \;=\; N_{k+1}, \;\; \langle \boldsymbol{y}, M\boldsymbol{y} \rangle \;=\; N_{k-1}.$$

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle^2 \geq \; \langle \boldsymbol{x}, M\boldsymbol{y} \rangle \, \langle \boldsymbol{y}, M\boldsymbol{y} \rangle \quad (\text{Hyp})$$

then implies

$$N_k^2 \;\geq\; N_{k+1} \, N_{k-1}.$$

Extension atlas

Recap: Atlas definition

A combinatorial atlas is a collection of $d \times d$ nonnegative symmetric matrices and vector:

$$M_0, M_1, \ldots, M_d \ \in \ \mathbb{R}_{\geq 0}^{d \times d}, \qquad \boldsymbol{h} \in \mathbb{R}_{\geq 0}^{d}.$$

M_0 is the parent of the atlas.

$M_1, \ldots, M_d$ are the children of the atlas.

We would want $M_0, \ldots, M_d$ to satisfy (Hyp).

Extension atlas

Fix $t \in (0,1)$ and $d := m$ and $k \in \{3, \ldots, n-1\}$.

Extension atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ is given by

$$M_0 := t\, \mathrm{E}[\mathcal{P}, k] + (1-t)\, \mathrm{E}[\mathcal{P}, k-1];$$
$$M_i := \mathrm{E}[\mathcal{P} - x_i, k-1] \quad (i \in [d]);$$
$$\boldsymbol{h} := (\underbrace{t, \ldots, t}_{\ell}, \underbrace{1-t, \ldots, 1-t}_{m-\ell});$$

where $\mathcal{P} - x :=$ poset $\mathcal{P}$ with x removed.

We will show that $M_0, \ldots, M_d$ satisfy (Hyp).

Recap: Children-to-parent principle

Theorem (Theorem 5.2 (C.-Pak 24))
Let atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ *satisfies* (Inh), (T-Inv), (Irr), (h-Pos), ~~(Dec)~~, *and* (Proj), (K-Non). *Then*

$M_1, \cdots, M_d$ *satisfy* (Hyp) $\Longrightarrow$ M_0 *satisfies* (Hyp).

Thus our strategy becomes:

- Assume $M_1, \ldots, M_d$ satisfy (Hyp) (induction);
- Verify (Inh), (T-Inv), ..., (K-Non);
- $\Longrightarrow$ M_0 satisfies (Hyp).

Proof of extension matrix theorem

Proof of extension matrix theorem, part 1

We will use induction on n. Base case is $n = 3$.

Then $\mathrm{E}[\mathcal{P}, k]$ is reduced to one of these matrices:

$$\begin{bmatrix}1\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}, \quad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{bmatrix}.$$

All these matrices satisfy (OPE), and thus (Hyp).

Proof of extension matrix theorem, part 2

Assume $n \geq 4$. Let $k \in \{3, \ldots, n-1\}$.

Then $M_1, \ldots, M_d$ satisfy (Hyp) by induction.

Children-parent-principle $\implies$ M_0 satisfies (Hyp).

$\iff t\,\mathrm{E}[\mathcal{P}, k] + (1-t)\,\mathrm{E}[\mathcal{P}, k-1]$ satisfies (Hyp).

- $t \to 1 \implies \mathrm{E}[\mathcal{P}, k]$ satisfies (Hyp),
  Thus $\mathrm{E}[\mathcal{P}, 3], \ldots, \mathrm{E}[\mathcal{P}, n-1]$ satisfy (Hyp).
- $t \to 0 \implies \mathrm{E}[\mathcal{P}, k-1]$ satisfies (Hyp),
  Thus $\mathrm{E}[\mathcal{P}, 2]$ also satisfies (Hyp). $\square$

What we have shown

Fix $z \in \mathcal{P}$. Let

$$N_k := \begin{array}{c}\text{number of linear extensions with} \\ z \text{ being } k\text{-th smallest.}\end{array}$$

Theorem (Stanley '81)

For every poset $\mathcal{P}$ *and* $k \in \{2, \ldots, n-1\}$,

$$N_k{}^2 \ \geq \ N_{k+1}\, N_{k-1}.$$

What we have shown

Fix $z \in \mathcal{P}$. Let

$$N_k := \begin{array}{c}\text{number of linear extensions with} \\ z \text{ being } k\text{-th smallest.}\end{array}$$

Theorem (Stanley '81)

For every poset $\mathcal{P}$ *and* $k \in \{2, \ldots, n-1\}$,

$$N_k^{\,2} \geq N_{k+1}\, N_{k-1}.$$

We will now discuss Stanley's equality problem.

Stanley's equality problem

Stanley's equality problem

Question (Stanley '81)

Find equality condition for [Stanley's inequality].

Quote (Gardner '02)

If inequalities are silver currency in mathematics, those that come along with precise equality conditions are gold.

Stanley's equality theorem (simple case)

Theorem (Shenfeld-van Handel '23)

Suppose $k \in \{2, \ldots, n-1\}$ *and* $N_k > 0$. *Then*

$$N_k^2 = N_{k+1}\, N_{k-1}$$

if and only if

$$|\mathcal{P}_{<x}| > k \qquad \text{for all } x \in \mathcal{P}_{>z},$$

$$|\mathcal{P}_{>x}| > |\mathcal{P}| - k + 1 \quad \text{for all } x \in \mathcal{P}_{<z},$$

where $\mathcal{P}_{<x} :=$ *set of* $y \in \mathcal{P}$ *with* $y < x$.

Proof used classifications of extremals of Aleksandrov-Fenchel inequality for convex polytopes.

Stanley's equality theorem (simple case)

Theorem (Shenfeld-van Handel '23)

Suppose $k \in \{2, \ldots, n-1\}$ *and* $N_k > 0$. *Then*

$$N_k^2 = N_{k+1} \, N_{k-1}$$

if and only if

$$|\mathcal{P}_{<x}| > k \qquad \text{for all } x \in \mathcal{P}_{>z},$$

$$|\mathcal{P}_{>x}| > |\mathcal{P}| - k + 1 \quad \text{for all } x \in \mathcal{P}_{<z},$$

where $\mathcal{P}_{<x} :=$ *set of* $y \in \mathcal{P}$ *with* $y < x$.

This is a combinatorial condition, and can be checked in $O(n^2)$ steps.

Stanley's equality theorem (simple case)

Theorem (Shenfeld-van Handel '23)

Suppose $k \in \{2, \ldots, n-1\}$ *and* $N_k > 0$. *Then*

$$N_k^2 = N_{k+1}\, N_{k-1}$$

if and only if

$$|\mathcal{P}_{<x}| > k \qquad \text{for all } x \in \mathcal{P}_{>z},$$

$$|\mathcal{P}_{>x}| > |\mathcal{P}| - k + 1 \quad \text{for all } x \in \mathcal{P}_{<z},$$

where $\mathcal{P}_{<x} :=$ *set of* $y \in \mathcal{P}$ *with* $y < x$.

We will present a new proof using atlas method.

Proving equality conditions using atlas

Hyperbolic equality property

Triplet $(M, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{r\times r} \times \mathbb{R}^r \times \mathbb{R}^r$ satisfies (H-Equ) if

$$\langle \boldsymbol{x}, M\boldsymbol{y}\rangle^2 = \langle \boldsymbol{x}, M\boldsymbol{x}\rangle \langle \boldsymbol{y}, M\boldsymbol{y}\rangle.$$

Stanley's equality and (H-Equ)

Let

$$M := \mathrm{E}[\mathcal{P}, k], \quad \boldsymbol{x} := (\mathbf{1}^{\ell}, \mathbf{0}^{m-\ell}), \quad \boldsymbol{y} := (\mathbf{0}^{\ell}, \mathbf{1}^{m-\ell}).$$

Then

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle = N_k, \quad \langle \boldsymbol{x}, M\boldsymbol{x} \rangle = N_{k+1}, \quad \langle \boldsymbol{y}, M\boldsymbol{y} \rangle = N_{k-1}.$$

$$N_k^2 = N_{k+1} N_{k-1} \qquad \text{if and only if}$$

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle^2 = \langle \boldsymbol{x}, M\boldsymbol{y} \rangle \langle \boldsymbol{y}, M\boldsymbol{y} \rangle \quad \text{(H-Equ)}.$$

Stanley's equality theorem thus reduces to understanding (H-Equ).

Parent-to-children principle

Theorem (Theorem 7.1 (C.-Pak 24))

Let atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ *satisfies* (Inh), (T-Inv), (Irr), (Proj), (K-Non), *and* (Hyp). *Then*

$$(M_0, \boldsymbol{x}, \boldsymbol{y}) \text{ satisfies (H-Equ)}$$
$$\Longrightarrow (M_i, \boldsymbol{x}, \boldsymbol{y}) \text{ satisfies (H-Equ)} \quad \text{if } h_i > 0.$$

We will use (H-Equ) for children matrices

to get combinatorial equality condition.

Proof of Stanley's equality theorem

Stanley's equality theorem (simple case)

Theorem

Suppose $k \in \{2, \ldots, n-1\}$ *and* $N_k > 0$. *Then*

$$N_k^2 = N_{k+1} N_{k-1}$$

if and only if

$$|\mathcal{P}_{<x}| > k \qquad \text{for all } x \in \mathcal{P}_{>z},$$

$$|\mathcal{P}_{>x}| > |\mathcal{P}| - k + 1 \qquad \text{for all } x \in \mathcal{P}_{<z},$$

where $\mathcal{P}_{<x} :=$ *set of* $y \in \mathcal{P}$ *with* $y < x$.

We will only prove $\Longrightarrow$ direction,

as $\Longleftarrow$ direction is straightforward.

Proof of Stanley's equality theorem, part 1

We will use induction on n. Base case is $n = 3$.

Then there are eleven possibilities for $(\mathcal{P}, k, z)$.

Validity of theorem is confirmed for all of them.

For $n \geq 4$, we proceed with atlas method.

Proof of Stanley's equality theorem, part 2

Let $n \geq 4$ and $d := m$ and $k \in \{3, \ldots, n-1\}$.

Our atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ is given by

$$\begin{aligned} M_0 &:= \mathrm{E}[\mathcal{P}, k]; \\ M_i &:= \mathrm{E}[\mathcal{P} - x_i, k-1] \quad (i \in [d]); \\ \boldsymbol{h} &:= (\underbrace{1, \ldots, 1}_{\ell}, \underbrace{0, \ldots, 0}_{m-\ell}). \end{aligned}$$

This is extension atlas with $t = 1$.

Proof of Stanley's equality theorem, part 3

$$\text{Let} \quad \boldsymbol{x} \; := (\mathbf{1}^{\ell}, \mathbf{0}^{m-\ell}), \quad \boldsymbol{y} \; := (\mathbf{0}^{\ell}, \mathbf{1}^{m-\ell}).$$

Then

$$N_k^2 = N_{k+1} N_{k-1} \qquad \text{implies}$$

$$\langle \boldsymbol{x}, M_0 \boldsymbol{y} \rangle^2 = \langle \boldsymbol{x}, M_0 \boldsymbol{y} \rangle \langle \boldsymbol{y}, M_0 \boldsymbol{y} \rangle \qquad \text{(H-Equ)}.$$

Parent-children principle then implies,

$$\langle \boldsymbol{x}, M_i \boldsymbol{y} \rangle^2 = \langle \boldsymbol{x}, M_i \boldsymbol{y} \rangle \langle \boldsymbol{y}, M_i \boldsymbol{y} \rangle \qquad \text{(H-Equ)}$$

for all $i \in [\ell]$.

Proof of Stanley's equality theorem, part 4

Recall $x_1, \ldots, x_\ell$ are minimal elements of $\mathcal{P}$.

Let $\mathcal{P}^{(i)} := \mathcal{P} - x_i$. Previous slide implies

$$(N_{\boldsymbol{k-1}}^{(i)})^2 = N_{\boldsymbol{k}}^{(i)} N_{\boldsymbol{k-2}}^{(i)} \qquad \text{for all } \; i \in [\ell],$$

where $N_k^{(i)}$:= number of linear extensions of $\mathcal{P}^{(i)}$ with z being k-th smallest.

Induction then implies

$$|\,\mathcal{P}_{>x}^{(i)}\,| > |\,\mathcal{P}^{(i)}\,| - (\boldsymbol{k-1}) + 1 \quad \text{for all } \; x \in \mathcal{P}_{<z}^{(i)}.$$

Proof of Stanley's equality theorem, part 5

Thus, for $x \in \mathcal{P}_{<z}$,

$$
\begin{aligned}
|\,\mathcal{P}_{>x}\,| \;&=\; |\,\mathcal{P}^{(i)}_{>x}\,| && \text{(minimality of } x_i\text{)}\\
&>\; |\,\mathcal{P}^{(i)}\,| - (k-1) + 1 && \text{(previous slide)}\\
&=\; |\,\mathcal{P}\,| - k + 1. && \text{(as } \mathcal{P}^{(i)} := \mathcal{P} - x_i\text{)}
\end{aligned}
$$

Hence we conclude

$$|\,\mathcal{P}_{>x}\,| > |\,\mathcal{P}\,| - k + 1 \quad \text{for all } \; x \in \mathcal{P}_{<z}\,.$$

Applying analogous dual argument,

$$|\,\mathcal{P}_{<x}\,| > k \qquad\qquad \text{for all } \; x \in \mathcal{P}_{>z}\,. \qquad \square$$

What we have shown

Theorem

Suppose $k \in \{2, \ldots, n-1\}$ *and* $N_k > 0$. *Then*

$$N_k^2 = N_{k+1}\, N_{k-1}$$

if and only if

$$|\,\mathcal{P}_{<x}\,| > k \qquad \text{for all} \;\; x \in \mathcal{P}_{>z},$$

$$|\,\mathcal{P}_{>x}\,| > |\,\mathcal{P}\,| - k + 1 \quad \text{for all} \;\; x \in \mathcal{P}_{<z},$$

where $\mathcal{P}_{<x} :=$ *set of* $y \in \mathcal{P}$ *with* $y < x$.

Approximate independence

Approximate independence for matchings

For graph G, let

$$m(G) \ := \ \text{number of matchings of } G.$$

Theorem (Kahn '00)

For distinct vertices x, y of G,

$$\frac{1}{2} \ \leq \ \frac{m(G)\ m(G-x-y)}{m(G-x)\ m(G-y)} \ \leq \ 2.$$

Quote (Kahn '00)

While we cannot expect exact independence, we do have considerable approximate independence.

Approximate independence for matroids

For matroid $\mathcal{M}$, let

$$B(\mathcal{M}) \;:=\; \text{number of bases of } \mathcal{M}.$$

Theorem (Huh–Schröter–Wang '22)

For distinct elements x, y of $\mathcal{M}$,

$$\frac{B(\mathcal{M})\, B(\mathcal{M}-x-y)}{B(\mathcal{M}-x)\, B(\mathcal{M}-y)} \;\leq\; 2\big(1-\tfrac{1}{d}\big).$$

None of the known matroids have ratios larger than $\frac{8}{7}$.

Approximate independence for extensions

For poset $\mathcal{P}$ with n elements, let

$$e(\mathcal{P}) \ := \ \text{number of linear extensions of } \mathcal{P}.$$

Theorem (Fishburn '84, C.-Pak '22)

For distinct minimal elements x, y of $\mathcal{P}$,

$$\frac{n}{n-1} \ \leq \ \frac{e(\mathcal{P})\ e(\mathcal{P}-x-y)}{e(\mathcal{P}-x)\ e(\mathcal{P}-y)} \ \leq \ 2.$$

Lower bound was proved by FKG inequality,

and is tight for antichains.

Approximate independence for extensions

For poset $\mathcal{P}$ with n elements, let

$$e(\mathcal{P}) \; := \; \text{number of linear extensions of } \mathcal{P}.$$

Theorem (Fishburn '84, C.-Pak '22)

For distinct minimal elements x, y of $\mathcal{P}$,

$$\frac{n}{n-1} \;\leq\; \frac{e(\mathcal{P})\, e(\mathcal{P}-x-y)}{e(\mathcal{P}-x)\, e(\mathcal{P}-y)} \;\leq\; 2.$$

Upper bound was proved by atlas method,

and is tight for antichains of two elements.

Proof of approximate independence for extensions (upper bound)

A consequence of (Hyp)

Lemma (Lemma 5.5, C.-Pak '22)

Let M be a matrix that satisfies (Hyp). Then, for all nonnegative vectors $\mathbf{v}, \mathbf{x}, \mathbf{y}$,

$$\langle \mathbf{v}, M\mathbf{v} \rangle \langle \mathbf{x}, M\mathbf{y} \rangle \ \leq \ 2 \langle \mathbf{v}, M\mathbf{x} \rangle \langle \mathbf{v}, M\mathbf{y} \rangle. \qquad \text{(2-Ind)}$$

Proof of approximate independence theorem, part 1

Let $\mathcal{P}' := \mathcal{P} + z$, where z is incomparable to all other elements of $\mathcal{P}$.

Let $x_1, \ldots, x_\ell$ be minimal elements of $\mathcal{P}'$.

W.l.o.g. $x = x_1$ and $y = x_2$ and $k \in \{2, \ldots, n-1\}$.

Set

$$M := \mathrm{E}[\mathcal{P}', k], \quad \boldsymbol{v} := (\mathbf{0}^\ell, \mathbf{1}^{m-\ell}),$$

$$\boldsymbol{x} := (1, 0, \ldots, 0), \quad \boldsymbol{y} := (0, 1, 0 \ldots, 0).$$

Proof of approximate independence theorem, part 2

Then

$$\langle \boldsymbol{v}, M\boldsymbol{v}\rangle = \text{number of extensions } \omega \text{ of } \mathcal{P}' \text{ with } \omega_{k-1}{=}\mathrm{z}.$$

$$= e(\mathcal{P}).$$

$$\langle \boldsymbol{v}, M\boldsymbol{x}\rangle = \text{number of extensions } \omega \text{ of } \mathcal{P}' \text{ with } \omega_k = z \text{ and } \omega_1 = x$$

$$= e(\mathcal{P} - x)$$

$$\langle \boldsymbol{v}, M\boldsymbol{y}\rangle = \text{number of extensions } \omega \text{ of } \mathcal{P}' \text{ with } \omega_k = z \text{ and } \omega_1 = y$$

$$= e(\mathcal{P} - y).$$

Proof of approximate independence theorem, part 3

And

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle \;=\; \text{number of extensions of } \mathcal{P}' \text{ with with } \omega_{k+1} = z \text{ and } \omega_1 = x \text{ and } \omega_2 = y$$

$$= \; e(\mathcal{P} - x - y).$$

$$\langle \boldsymbol{v}, M\boldsymbol{v} \rangle \langle \boldsymbol{x}, M\boldsymbol{y} \rangle \;\le\; 2\,\langle \boldsymbol{v}, M\boldsymbol{x} \rangle\,\langle \boldsymbol{v}, M\boldsymbol{y} \rangle \quad (\text{2-Ind})$$

then implies

$$e(\mathcal{P})\,e(\mathcal{P} - x - y) \;\le\; 2\,e(\mathcal{P} - x)\,e(\mathcal{P} - y). \quad \square$$

What we have done

For poset $\mathcal{P}$ with n elements, let

$$e(\mathcal{P}) \;:=\; \text{number of linear extensions of } \mathcal{P}.$$

Theorem (Fishburn '84, C.-Pak '22)

For distinct minimal elements x, y of $\mathcal{P}$,

$$\frac{n}{n-1} \;\leq\; \frac{e(\mathcal{P})\; e(\mathcal{P}-x-y)}{e(\mathcal{P}-x)\; e(\mathcal{P}-y)} \;\leq\; 2.$$

Next episode preview

What we did today

Theorem (Stanley's inequality, simple form)

Suppose $k \in \{2, \ldots, n-1\}$ *and* $N_k > 0$. *Then*

$$N_k^2 \geq N_{k+1} \, N_{k-1},$$

with equality *if and only if*

$$|\mathcal{P}_{<x}| > k \qquad \text{for all } x \in \mathcal{P}_{>z},$$

$$|\mathcal{P}_{>x}| > |\mathcal{P}| - k + 1 \qquad \text{for all } x \in \mathcal{P}_{<z},$$

where $\mathcal{P}_{<x} :=$ *set of* $y \in \mathcal{P}$ *with* $y < x$.

The complexity aspect of full version of Stanley's inequality will be discussed in **Day 4**.

SEE YOU NEXT CLASS!

References: www.arxiv.org/abs/2203.01533

www.arxiv.org/abs/2211.16637

Webpage: www.math.rutgers.edu/~sc2518/

Email: sweehong.chan@rutgers.edu