Log-concave Poset Inequalities

Day 2: Improvement to Mason's Conjecture

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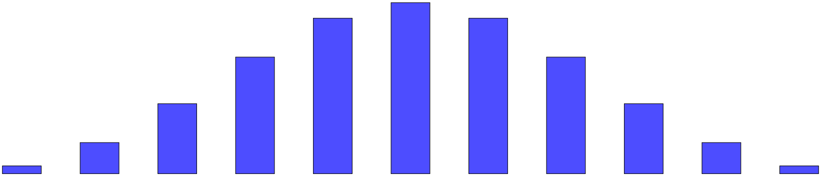
joint with Igor Pak



Recap: Matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphical matroids

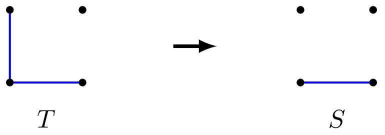
- X = edges of a graph G,
- $\mathcal{I}$ = forests in G.

Realizable matroids

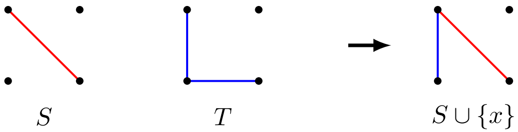
- X = finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I}$ = sets of linearly independent vectors.

Recap: Axioms of matroids

- (Hereditary) $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.



- (Exchange) If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



Recap: Matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

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- X = edges of a graph G,
- $\mathcal{I}$ = forests in G.

Realizable matroids

- X = finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I}$ = sets of linearly independent vectors.

Recap: Mason's Conjecture (1972)

Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $n = |X|$.

$$a_k := \text{no. of independent sets of size } k.$$

It was conjectured that, for $0 < k < n$:

(1) ${a_k}^2 \ \geq \ a_{k+1}\, a_{k-1}$;

(2) ${a_k}^2 \ \geq \ \left(1 + \frac{1}{k}\right) a_{k+1}\, a_{k-1}$;

(3) ${a_k}^2 \ \geq \ \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) a_{k+1}\, a_{k-1}$.

We previously proved Mason (2)

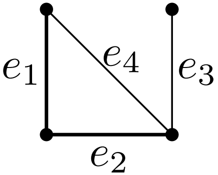
for graphic matroids in Day 1.

Proof of Mason (2)
for all matroids

Independent words

A word $\omega = \omega_1 \cdots \omega_k \in X^*$ is a independent word if

$$\{\omega_1, \ldots, \omega_k\} \text{ is an independent set.}$$



$e_1 e_3 e_2$ and $e_1 e_2 e_3$ are independent words of $\mathcal{M}_G$.

$e_1 e_2 e_4$ and $e_2 e_4 e_2$ are NOT independent words.

Independent matrix (for Mason (2))

Let $X = \{x_1, \dots, x_n\}$ and $0 < k < n$.

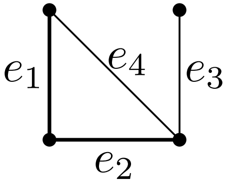
Independent matrix $\mathrm{I}_2[\mathcal{M}, k]$ is $(n+1) \times (n+1)$ matrix where, for $i, j \in [n]$:

$$(\mathrm{I}_2[\mathcal{M}, k])_{i,j} := \begin{array}{l} \text{no. of ind. words of length } k+1 \\ \text{starts with } x_i\text{, ends with } x_j \end{array} ;$$

$$(\mathrm{I}_2[\mathcal{M}, k])_{i,n+1} := \begin{array}{l} \text{no. of ind. words of length } k \\ \text{starts with } x_i \end{array} ;$$

$$(\mathrm{I}_2[\mathcal{M}, k])_{n+1,n+1} := \begin{array}{l} \text{no. of ind. words of length} \\ k-1 \end{array} .$$

Example: Independent matrix, $k = 2$



G

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 3 \\ 1 & 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 1 & 2 & 0 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{bmatrix}$$

$\mathrm{I}_2[\mathcal{M}_G, 2]$

$$(\mathrm{I}_2[\mathcal{M}_G, 2])_{2,3} := |\{e_2 e_1 e_3,\ e_2 e_4 e_3\}| = 2;$$

$$(\mathrm{I}_2[\mathcal{M}_G, 2])_{2,5} := |\{e_2 e_1,\ e_2 e_3,\ e_2 e_4\}| = 3;$$

$$(\mathrm{I}_2[\mathcal{M}_G, 2])_{5,5} := |\{e_1,\ e_2,\ e_3,\ e_4\}| = 4.$$

Recap: Hyperbolic property

M has hyperbolic property (Hyp) if

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, M\boldsymbol{x} \rangle \langle \boldsymbol{y}, M\boldsymbol{y} \rangle$$

for every $\boldsymbol{x} \in \mathbb{R}^r$ and $\boldsymbol{y} \in \mathbb{R}^r_{\geq 0}$.

M satisfies (OPE) if

$\boldsymbol{M}$ has at most one positive eigenvalue.

Lemma (Lemma 3.5 (C.–Pak 22))

M satisfies (Hyp) $\iff$ *M satisfies* (OPE).

Independent matrix theorem

Theorem

For every matroid $\mathcal{M}$ and $0 < k < n$,

$$\textit{The matrix } \mathrm{I}_2[\mathcal{M}, k] \textit{ satisfies } (\text{Hyp}).$$

This theorem implies Mason (2).

Independent matrix theorem implies Mason (2)

Let

$$M := \mathrm{I}_2[\mathcal{M}, k], \;\; \boldsymbol{x} := (1, \ldots, 1, 0), \;\; \boldsymbol{y} := (0, \ldots, 0, 1).$$

Then

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle = k!\, a_k, \quad \langle \boldsymbol{x}, M\boldsymbol{x} \rangle = (k+1)!\, a_{k+1},$$

$$\langle \boldsymbol{y}, M\boldsymbol{y} \rangle = (k-1)!\, a_{k-1}.$$

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, M\boldsymbol{y} \rangle \langle \boldsymbol{y}, M\boldsymbol{y} \rangle \quad \text{(Hyp)}$$

then implies

$$a_k^2 \geq \left(1 + \tfrac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

Independent atlas

Recap: Atlas definition

A combinatorial atlas is a collection of $d \times d$ nonnegative symmetric matrices and vector:

$$M_0, M_1, \ldots, M_d \in \mathbb{R}_{\geq 0}^{d \times d}, \qquad \boldsymbol{h} \in \mathbb{R}_{\geq 0}^{d}.$$

M_0 is the parent of the atlas.

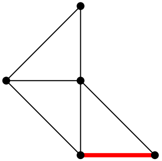
$M_1, \ldots, M_d$ are the children of the atlas.

We would want $M_0, \ldots, M_d$ to satisfy hyperbolic property.

Matroid contraction

The contraction of element $x \in X$ of matroid $\mathcal{M}$
is matroid $\mathcal{M}/x := (X', \mathcal{I}')$ where

$$X' := X \setminus \{x\}; \quad \mathcal{I}' := \{ S \subseteq X' : S \cup \{x\} \in \mathcal{I} \}.$$



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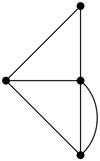
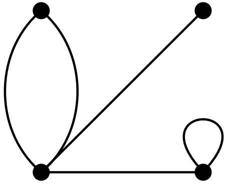


G/e

Matroid: loops and parallel elements

A loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops $x, y \in X$ are parallel if $\{x, y\} \notin \mathcal{I}$.



A matroid is simple if it has
no loops and parallel elements.

Independent atlas

Fix $t \in (0,1)$ and $d := n+1$ and $2 \le k < n$.

Independent atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ is given by

$$\begin{aligned} M_0 &:= t\,\mathrm{I}_2[\mathcal{M}, k] + (1-t)\,\mathrm{I}_2[\mathcal{M}, k-1]; \\ M_i &:= \mathrm{I}_2[\mathcal{M}/x_i, k-1] \quad (i \in [n]); \\ M_d &:= \mathrm{I}_2[\mathcal{M}, k-1]; \\ \boldsymbol{h} &:= (t, \ldots, t, 1-t). \end{aligned}$$

We will show that $M_0, \ldots, M_d$ satisfy (Hyp).

Recap: Children-to-parent principle

Theorem (Theorem 3.4 (C.-Pak 22))
Let atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ *satisfies* (Inh), (T-Inv), (Dec), (Irr), *and* (h-Pos). *Then*

$$M_1, \cdots, M_d \text{ satisfy (Hyp)} \implies M_0 \text{ satisfies (Hyp)}.$$

Thus our strategy becomes:

- Assume $M_1, \ldots, M_d$ satisfy (Hyp) (induction);
- Verify (Inh), (T-Inv), (Dec), (Irr), (h-Pos);
- $\implies$ M_0 satisfies (Hyp).

Proof of independent matrix theorem

Proof of independent matrix theorem

We will use induction on k. Base case is $k = 1$.

W.l.o.g. $\mathcal{M}$ has no loops or parallel elements. Then

$$\mathrm{I}_2[\mathcal{M}, 1] = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

is $d \times d$ matrix with eigenvalues approximately

$$d - 1 + \tfrac{1}{d-1}, -1, \ldots, -1, -\tfrac{1}{d-1},$$

so $\mathrm{I}_2[\mathcal{M}, 1]$ satisfies (OPE), and thus (Hyp).

Proof of independent matrix theorem

Assume $k \geq 2$.

Then $M_1, \dots, M_d$ satisfy (Hyp) by induction.

Children-parent-principle $\implies$ M_0 satisfies (Hyp).

$\iff t\,\mathrm{I}_2[\mathcal{M}, k] + (1-t)\,\mathrm{I}_2[\mathcal{M}, k-1]$ satisfies (Hyp).

$t \to 1 \implies \mathrm{I}_2[\mathcal{M}, k]$ satisfies (Hyp). $\square$

Recap complete

Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $n = |X|$.

$$a_k := \text{no. of independent sets of size } k.$$

We have shown Mason (2):

$$a_k{}^2 \;\geq\; \left(1 + \tfrac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

Recap complete

Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $n = |X|$.

$$a_k := \text{no. of independent sets of size } k.$$

We have shown Mason (2):

$$a_k{}^2 \;\geq\; \left(1 + \tfrac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

We now show how to improve to Mason (3):

$$a_k{}^2 \;\geq\; \left(1 + \tfrac{1}{k}\right) \left(1 + \tfrac{1}{n-k}\right) a_{k+1}\, a_{k-1}.$$

Which part of proof can be improved?

Looking back to our proof

Recall the $d \times d$ matrix:

$$\mathrm{I}_2[\mathcal{M}, 1] = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

satisfies (OPE) with eigenvalues approximately:

$$d - 1 + \tfrac{1}{d-1}, -1, \ldots, -1, -\tfrac{1}{d-1},$$

There are rooms for improvements here!

Room for improvement

Changing $\mathrm{I}_2[\mathcal{M}, 1]$ to $d \times d$ matrix:

$$\begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & \mathbf{1} + \varepsilon \end{bmatrix}$$

has eigenvalues approximately

$$d - 1 + \tfrac{1}{d-1} - O(\varepsilon), -1, \ldots, -1, -\tfrac{1}{d-1} + O(\varepsilon),$$

still satisfies (OPE) for small ε and implies

$${a_k}^2 \;\geq\; \left(1 + \tfrac{1}{k}\right)\left(\mathbf{1} + \varepsilon\right) a_{k+1}\, a_{k-1} \quad (k = 1).$$

Independent matrix (3)

Independent matrix $\mathrm{I}_3[\mathcal{M}, k]$ is $(n+1) \times (n+1)$ matrix where, for $i, j \in [n]$:

$(\mathrm{I}_3[\mathcal{M}, k])_{i,j} \; := \; \boldsymbol{(n-k-1)!}$ no. of ind. words of length $k+1$, starts with x_i, ends with x_j;

$(\mathrm{I}_3[\mathcal{M}, k])_{i,n+1} \; := \; \boldsymbol{(n-k)!}$ no. of ind. words of length k, starts with x_i;

$(\mathrm{I}_3[\mathcal{M}, k])_{n+1,n+1} \; := \; \boldsymbol{(n-k+1)!}$ no. of ind. words of length $k-1$.

Independent matrix theorem (3)

Theorem

For every matroid $\mathcal{M}$ and $0 < k < n$,

$$\textit{The new matrix } \mathrm{I}_3[\mathcal{M}, k] \textit{ satisfies } (\mathrm{Hyp}).$$

This improved theorem implies Mason (3).

Independent matrix theorem (3) implies Mason (3)

$M := \mathrm{I}_3[\mathcal{M}, k]$, $\boldsymbol{x} := (1, \ldots, 1, 0)$, $\boldsymbol{y} := (0, \ldots, 0, 1)$.

Then

$$\begin{aligned} \langle \boldsymbol{x}, M\boldsymbol{y} \rangle &= k!\,(\boldsymbol{n-k})!\,a_k; \\ \langle \boldsymbol{x}, M\boldsymbol{x} \rangle &= (k+1)!\,(\boldsymbol{n-k-1})!\,a_{k+1}; \\ \langle \boldsymbol{y}, M\boldsymbol{y} \rangle &= (k-1)!\,.(\boldsymbol{n-k+1})!\,a_{k-1}. \end{aligned}$$

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, M\boldsymbol{y} \rangle \langle \boldsymbol{y}, M\boldsymbol{y} \rangle \quad (\text{Hyp})$$

then implies

$$a_k^2 \geq \left(1 + \frac{1}{k}\right)\left(\boldsymbol{1} + \frac{\boldsymbol{1}}{\boldsymbol{n-k}}\right) a_{k+1}\, a_{k-1}.$$

Proof of Mason (3)
independent matrix theorem

Proof of independent matrix theorem (3)

Let $k = 1$. W.l.o.g. $\mathcal{M}$ is simple matroid.

Then $\mathrm{I}_3[\mathcal{M}, 1]$ is $(n+1) \times (n+1)$ matrix:

$$\mathrm{I}_3[\mathcal{M}, 1] = \begin{bmatrix} 0 & (n-2)! & \cdots & (n-2)! & (n-1)! \\ (n-2)! & 0 & \cdots & (n-2)! & (n-1)! \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-2)! & (n-2)! & \cdots & 0 & (n-1)! \\ (n-1)! & (n-1)! & \cdots & (n-1)! & n! \end{bmatrix}.$$

The eigenvalues are

$$(n+1)(n-1)!, \mathbf{0}, -(n-2)!, \ldots, -(n-2)!.$$

so $\mathrm{I}_3[\mathcal{M}, 1]$ satisfies (OPE), and thus (Hyp).

Proof of independent matrix theorem (3)

Assume $k \geq 2$.

Fix $t \in (0,1)$ and $d := n+1$ and $2 \leq k < n$.

Independent atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ is given by

$$\begin{aligned} M_0 &:= t\, \mathrm{I}_3[\mathcal{M}, k] + (1-t)\, \mathrm{I}_3[\mathcal{M}, k-1]; \\ M_i &:= \mathrm{I}_3[\mathcal{M}/x_i, k-1] \quad (i \in [n]); \\ M_d &:= \mathrm{I}_3[\mathcal{M}, k-1]; \\ \boldsymbol{h} &:= (t, \ldots, t, 1-t). \end{aligned}$$

The definition of this atlas does not change.

Proof of independent matrix theorem (3)

Then $M_1, \ldots, M_d$ satisfy (Hyp) by induction.

Children-parent principle $\implies$ M_0 satisfies (Hyp).

$\iff t\, \mathrm{I}_3[\mathcal{M}, k] + (1-t)\, \mathrm{I}_3[\mathcal{M}, k-1]$ satisfies (Hyp).

$t \to 1 \implies \mathrm{I}_3[\mathcal{M}, k]$ satisfies (Hyp). $\square$

This part of the proof does not change.

What we have shown

Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $n = |X|$.

$a_k :=$ no. of independent sets of size k.

Theorem (Mason (3))

$$a_k{}^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) a_{k+1}\, a_{k-1}.$$

What we have shown

Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $n = |X|$.

$a_k :=$ no. of independent sets of size k.

Theorem (Mason (3))

$$a_k{}^2 \;\geq\; \left(1 + \tfrac{1}{k}\right)\left(1 + \tfrac{1}{n-k}\right) a_{k+1}\, a_{k-1}.$$

Next we show that this inequality can be improved **even further**.

Mason (4) for graphical matroids

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Theorem (C.-Pak)

For graphical matroid of connected graph $G = (V, E)$, and $\kappa = |V| - 2$,

$$(a_\kappa)^2 \;\geq\; \frac{3}{2}\left(1 + \frac{1}{\kappa}\right) a_{\kappa+1}\, a_{\kappa-1}.$$

Numerically better than Mason (3), because

$$\frac{3}{2} \;\geq\; 1 + \frac{1}{n-\kappa} \;=\; 1 + \frac{1}{|E| - |V| + 2}$$

for G that is not tree.

Comparison with Mason (3)

Mason (4) gives

$$\frac{(a_\kappa)^2}{a_{\kappa+1}\, a_{\kappa-1}} \quad \geq \quad \frac{3}{2} \qquad \text{when} \quad |E| - |V| \to \infty,$$

Meanwhile, Mason (3) only gives

$$\frac{(a_\kappa)^2}{a_{\kappa+1}\, a_{\kappa-1}} \quad \geq \quad 1 \qquad \text{when} \quad |E| - |V| \to \infty.$$

New bound is better numerically and asymptotically.

We will prove Mason (4) today using atlas method.

Which part of proof can be further improved?

Room for further improvement

Recall $(n+1) \times (n+1)$ matrix:

$$\mathrm{I}_3[\mathcal{M}, 1] = \begin{bmatrix} 0 & (n-2)! & \cdots & (n-2)! & (n-1)! \\ (n-2)! & 0 & \cdots & (n-2)! & (n-1)! \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-2)! & (n-2)! & \cdots & 0 & (n-1)! \\ (n-1)! & (n-1)! & \cdots & (n-1)! & n! \end{bmatrix}$$

has eigenvalues

$$(n+1)(n-1)!, \mathbf{0}, -(n-2)!, \ldots, -(n-2)!\,.$$

No room for improvement for general matroids.

But such matrix never occurs for **graphic matroids**.

Independent matrix for graphic Mason (4)

Let $E = \{e_1, \ldots, e_n\}$ and $\kappa = |V| - 2$.

Independent matrix $\mathrm{I}_4[\mathcal{M}_G, \kappa]$ is $(n+1) \times (n+1)$ matrix where, for $i, j \in [n]$:

$$(\mathrm{I}_4[\mathcal{M}_G, \kappa])_{i,j} \;:=\; \begin{array}{l}\text{no. of ind. words of length } \kappa + 1 \\ \text{starts with } e_i\text{, ends with } e_j\end{array};$$

$$(\mathrm{I}_4[\mathcal{M}_G, \kappa])_{i,n+1} \;:=\; \begin{array}{l}\text{no. of ind. words of length } \kappa \\ \text{starts with } e_i\end{array};$$

$$(\mathrm{I}_4[\mathcal{M}_G, \kappa])_{n+1,n+1} \;:=\; \mathbf{\frac{3}{2}} \times \begin{array}{l}\text{no. of ind. words of} \\ \text{length } \kappa - 1\end{array}.$$

Independent matrix theorem (4)

Theorem

For every graph $G = (V, E)$ and $\kappa = |V| - 2$,

$$\text{The new matrix } \mathrm{I}_4[\mathcal{M}_G, \kappa] \text{ satisfies (Hyp).}$$

This improved theorem implies graphic Mason (4).

Ind. matrix theorem (4) implies graphic Mason (4)

$M := \mathrm{I}_4[\mathcal{M}_G, \kappa]$, $\boldsymbol{x} := (1, \ldots, 1, 0)$, $\boldsymbol{y} := (0, \ldots, 0, 1)$.

Then

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle = \kappa!\, a_\kappa; \qquad \langle \boldsymbol{y}, M\boldsymbol{y} \rangle = (\kappa - 1)!\, a_{\kappa-1};$$

$$\langle \boldsymbol{x}, M\boldsymbol{x} \rangle = \frac{3}{2}\, (\kappa + 1)!\, a_{\kappa+1}.$$

$$\langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y} \rangle \langle \boldsymbol{y}, \boldsymbol{M}\boldsymbol{y} \rangle \quad (\text{Hyp})$$

then implies

$$a_k^2 \geq \frac{3}{2} \left(1 + \frac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

Proof of independent matrix theorem (4)

Proof of independent matrix theorem (4)

We prove by induction on $|V|$. Let $|V| = 3$.

W.l.o.g $G = (V, E)$ is simple graph.

Then $n = |E| \leq 3$, and $\mathrm{I}_4[\mathcal{M}_G, \kappa]$ is

$$\text{either} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & \frac{3}{2} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & \frac{3}{2} \end{bmatrix}$$

The eigenvalues are

$$(2.6\overline{9}, -1, -0.1\overline{9}) \quad \text{or} \quad (\tfrac{7}{2}, \mathbf{0}, -1, -1)$$

so $\mathrm{I}_4[\mathcal{M}_G, \kappa]$ satisfies (OPE), and thus (Hyp).

Proof of independent matrix theorem (4)

Assume $|V| \geq 4$.

Fix $t \in (0,1)$ and $d := |E| + 1$ and $\kappa := |V| - 2$.

Independent atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ is given by

$$\begin{aligned} M_0 &:= t\, \mathrm{I}_4[\mathcal{M}_G, \kappa] + (1-t)\, \mathrm{I}_2[\mathcal{M}_G, \kappa - 1]; \\ M_i &:= \mathrm{I}_4[\mathcal{M}_{G/e_i}, \kappa - 1] \quad (1 \leq i \leq |E|); \\ M_d &:= \mathrm{I}_2[\mathcal{M}_G, \kappa - 1]; \\ \boldsymbol{h} &:= (t, \ldots, t, 1-t). \end{aligned}$$

Proof of independent matrix theorem (4)

Then $M_1, \ldots, M_d$ satisfy (Hyp) by induction.

Children-parent principle $\implies$ M_0 satisfies (Hyp).

$\iff$ $t\,\mathrm{I}_4[\mathcal{M}_G, \kappa] + (1-t)\,\mathrm{I}_2[\mathcal{M}_G, \kappa - 1]$ satisfies (Hyp).

$t \to 1 \implies \mathrm{I}_4[\mathcal{M}_G, \kappa]$ satisfies (Hyp). $\square$

This part of the proof does not change.

What we have shown

Theorem (C.-Pak '24)
For graphical matroid of simple connected graph $G = (V, E)$, and $\kappa = |V| - 2$,

$$(a_\kappa)^2 \ \geq \ \frac{3}{2}\left(1 + \frac{1}{\kappa}\right) a_{\kappa+1}\, a_{\kappa-1},$$

with equality if and only if G is a cycle graph.

Naturally, this improvement can be generalized to all matroids (next slide).

Mason (4) for all matroids

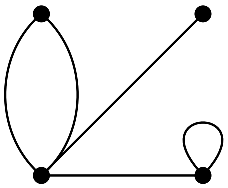
Parallel classes

A loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops $x, y \in X$ are parallel if $\{x, y\} \notin \mathcal{I}$.

Parallel equivalence relation: $x \sim y$ if $\{x, y\} \notin \mathcal{I}$.

Parallel class $=$ equivalence class of $\sim$.



Parallel number

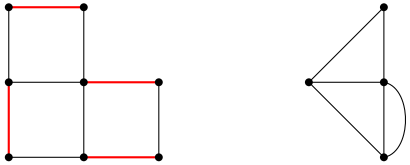
Contraction of $S \in \mathcal{I}$ is matroid $\mathcal{M}_S := (X_S, \mathcal{I}_S)$:

$$X_S \ := \ X \setminus S, \qquad \mathcal{I}_S \ := \ \{T \setminus S \,:\, S \subseteq T\}.$$

For $S \in \mathcal{I}$, the parallel number is

$\mathrm{prl}(S) \ := \$ number of parallel classes of $\mathcal{M}_S$

$= \$ no. of elements of $\mathcal{M}_S$ not counting multiplicity.

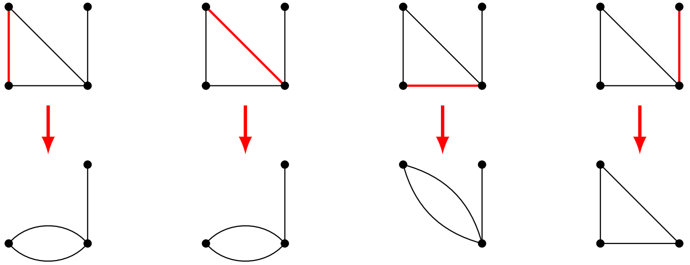


$$\mathrm{prl}(S) = 5$$

Parallel constant

The *k-th parallel constant* is

$$\mathrm{p}[\mathcal{M}, k] \;:=\; \max\{\mathrm{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$



$\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 2$ $\mathrm{prl}(S) = 3$

$\mathrm{p}[\mathcal{M}, 1] = 3$

Mason (4)

Theorem (Thm 1.10, C.-Pak '24)

For matroid $\mathcal{M}$ and $0 < k < n$,

$$a_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}[\mathcal{M}, k-1] - 1}\right) a_{k+1}\, a_{k-1}.$$

This refines Mason (3),

$$a_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) a_{k+1}\, a_{k-1},$$

since

$$\mathrm{p}[\mathcal{M}, k-1] \;\leq\; n - k + 1.$$

Improvement for different matroids

- For all matroids,

  $${a_k}^2 \;\geq\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) a_{k+1}\, a_{k-1}.$$

- Graphical matroids and $k = |V| - 2$,

  $${a_k}^2 \;\geq\; \left(1+\frac{1}{k}\right)\frac{3}{2}\, a_{k+1}\, a_{k-1}.$$

- Realizable matroids over $\mathbb{F}_q$,

  $${a_k}^2 \;\geq\; \left(1+\frac{1}{k}\right)\left(1+\frac{1}{q^{m-k+1}-2}\right) a_{k+1}\, a_{k-1}.$$

- (k, m, n)-Steiner system matroid,

  $${a_k}^2 \;\geq\; \left(1+\frac{1}{k}\right)\frac{n-k+1}{n-m}\, a_{k+1}\, a_{k-1}.$$

Mason (4)

Theorem (Thm 1.10, C.-Pak '24)

For matroid $\mathcal{M}$ *and* $0 < k < n$,

$$a_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}[\mathcal{M}, k-1] - 1}\right) a_{k+1}\, a_{k-1}.$$

Naturally, we have matching equality conditions for this inequality.

When is equality achieved?

- When $\mathcal{M}$ is free matroid, i.e. every subset of X is independent;
- When $\mathcal{M}$ is graphical matroid, with $G =$ cycle graph and $k = |V| - 2$;
- When $\mathcal{M}$ is realizable matroid, with $X := \{\textit{every } m\text{-dimensional vector over } \mathbb{F}_q\}$;
- When $\mathcal{M}$ is Steiner system matroid;
- $\cdots\cdots$ (running out of space) $\cdots\cdots$

Equality conditions for Mason (4)

Theorem (Thm 1.10, C.-Pak '24)

For matroid $\mathcal{M}$ *and* $0 < k < n$,

$$a_k{}^2 = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}[\mathcal{M}, k-1] - 1}\right) a_{k+1}\, a_{k-1}$$

if and only if *for all* $S \in \mathcal{I}$ *with* $|S| = k - 1$,

- $\mathcal{M}_S$ *has* $\mathrm{p}[\mathcal{M}, k-1]$ *parallel classes; and*
- *Every parallel class of* $\mathcal{M}_S$ *has same size.*

Equality conditions for Mason (4)

Theorem (Thm 1.10, C.-Pak '24)

For matroid $\mathcal{M}$ *and* $0 < k < n$,

$$a_k{}^2 = \left(1 + \tfrac{1}{k}\right)\left(1 + \tfrac{1}{\mathrm{p}[\mathcal{M},k-1]-1}\right) a_{k+1}\, a_{k-1}$$

if and only if *for all* $S \in \mathcal{I}$ *with* $|S| = k - 1$,

- $\mathcal{M}_S$ *has* $\mathrm{p}[\mathcal{M}, k-1]$ *parallel classes; and*
- *Every parallel class of* $\mathcal{M}_S$ *has same size.*

Quote (Dr. Strangelove '64)

How I Learned to Stop Worrying and Love

[*Convoluted Equality Conditions*].

Other applications of combinatorial atlas

We get log-concave inequalities and matching equality conditions for:

- Mason's inequality (refined) **Day 1–2**;
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined) **Day 3–4**;
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids;
- Stanley–Yan matroid basis inequality **Day 4**.

SEE YOU NEXT CLASS!

References: www.arxiv.org/abs/2110.10740

www.arxiv.org/abs/2203.01533

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