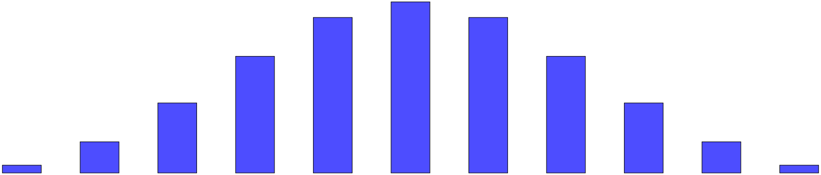
Log-concave Poset Inequalities

Day 1: Intro to Combinatorial Atlas

Swee Hong Chan

joint with Igor Pak



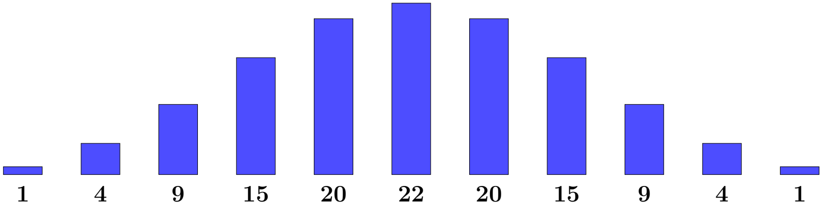
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \geq a_{k+1}\, a_{k-1} \qquad (1 < k < n).$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \leq \cdots \leq a_m \geq \cdots \geq a_n \quad \text{for some } 1 \leq m \leq n.$$



Example 1: binomial coefficients

$$a_k = \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\, a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example 2: permutations with k inversions

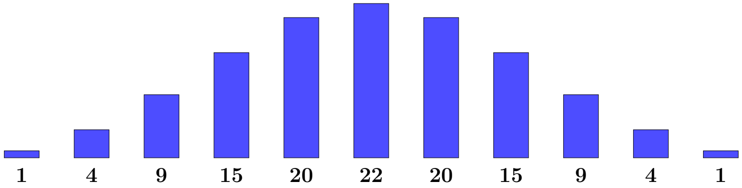
$$a_k \;=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; \prod_{i=1}^{n-1} (1 + q + q^2 + \ldots + q^i)$$

is a product of log-concave polynomials.

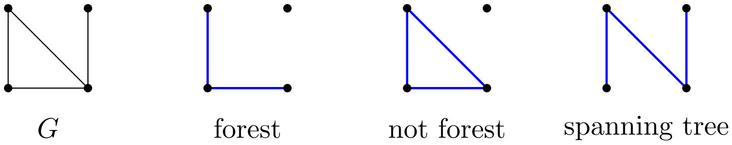


Example 3: forests of a graph

a_k = number of forests with k edges of graph G.

Forest is a subset of edges of G that has no cycles.

Log-concavity was conjectured for all matroids (Mason '72), and was proved through combinatorial Hodge theory (Huh '15).

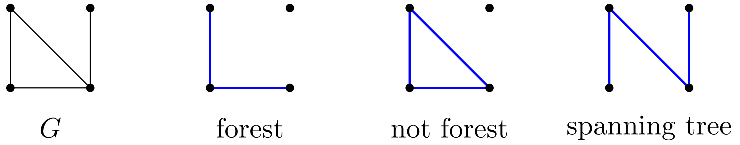


Example 3: forests of a graph

a_k = number of forests with k edges of graph G.

Forest is a subset of edges of G that has no cycles.

We will provide another proof of Mason's conjecture in this lecture series using combinatorial atlas.



Combinatorial atlas: version 1

Atlas v1: Definition

A combinatorial atlas is a collection of $d \times d$ nonnegative symmetric matrices and vector:

$$M_0, M_1, \ldots, M_d \;\in\; \mathbb{R}_{\geq 0}^{d \times d}, \qquad \boldsymbol{h} \in \mathbb{R}_{\geq 0}^{d}.$$

M_0 is the parent of the atlas.

$M_1, \ldots, M_d$ are the children of the atlas.

We would want $M_0, \ldots, M_d$ to satisfy hyperbolic property.

Hyperbolic property

M has hyperbolic property (Hyp) if

$$\langle \boldsymbol{x}, M\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, M\boldsymbol{x} \rangle \langle \boldsymbol{y}, M\boldsymbol{y} \rangle$$

for every $\boldsymbol{x} \in \mathbb{R}^r$ and $\boldsymbol{y} \in \mathbb{R}^r_{\geq 0}$.

M satisfies (OPE) if

$\boldsymbol{M}$ has at most one positive eigenvalue.

Lemma (Lemma 3.5 (C.–Pak 22))

M satisfies (Hyp) $\iff$ *M satisfies* (OPE).

Fantastic (Hyp) and where to find them

They are everywhere in mathematics:

Convex geometry: Alexandrov–Fenchel inequality.

Algebraic geometry: Hodge theory.

Combinatorics: Lorentzian polynomials.

How does (Hyp) imply log-concavity?

Input: $a_{k-1}, a_k, a_{k+1} \in \mathbb{R}$.

Find: $M \in \mathbb{R}_{\geq 0}^{d \times d}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_{\geq 0}^{d}$ such that

$$a_k = \langle \boldsymbol{x}, M\boldsymbol{y} \rangle, \quad a_{k+1} = \langle \boldsymbol{x}, M\boldsymbol{x} \rangle, \quad a_{k-1} = \langle \boldsymbol{y}, M\boldsymbol{y} \rangle.$$

$$\langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y} \rangle^2 \geq \langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y} \rangle \langle \boldsymbol{y}, \boldsymbol{M}\boldsymbol{y} \rangle \quad \text{(Hyp)}$$

then implies

$$a_k^2 \geq a_{k+1} a_{k-1} \quad \text{(log-concave ineq.)}$$

Mason’s (graphic) conjecture

Mason's graphic conjecture

Let $G = (V, E)$ be a graph.

$$a_k \; = \; \text{number of forests of } G \text{ with } k \text{ edges.}$$

Forest is a subset of edges of G that has no cycles.

Conjecture (Mason '72)

For $0 < k < |V|$,

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1}.$$

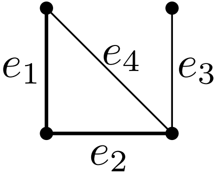
We will in fact show that

$$a_k^2 \;\geq\; \left(1 + \tfrac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

Forest words

A word $\omega = \omega_1 \cdots \omega_k \in E^*$ is a forest word if

$$\{\omega_1, \ldots, \omega_k\} \text{ is the edge set of a forest.}$$



$e_1 e_3 e_2$ and $e_1 e_3 e_2$ are forest words.

$e_1 e_2 e_4$ and $e_2 e_4 e_2$ are NOT forest words.

Forest matrix

Let $E = \{e_1, \ldots, e_m\}$ and $0 < k < |V|$.

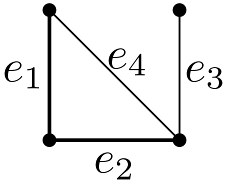
Forest matrix $F[G,k]$ is $(m+1) \times (m+1)$ matrix where, for $i,j \in [m]$:

$$(F[G,k])_{i,j} := \begin{array}{l}\text{no. of forest words of length } k+1 \\ \text{starts with } e_i, \text{ ends with } e_j\end{array};$$

$$(F[G,k])_{i,m+1} := \begin{array}{l}\text{no. of forest words of length } k \\ \text{starts with } e_i\end{array};$$

$$(F[G,k])_{m+1,m+1} := \begin{array}{l}\text{no. of forest words of length} \\ k-1\end{array}.$$

Example: Forest matrix, $k = 2$



G

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 3 \\ 1 & 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 1 & 2 & 0 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{bmatrix}$$

$F[G, 2]$

$$(F[G,2])_{2,3} := |\{e_2 e_1 e_3,\ e_2 e_4 e_3\}| = 2;$$

$$(F[G,2])_{2,5} := |\{e_2 e_1,\ e_2 e_3,\ e_2 e_4\}| = 3;$$

$$(F[G,2])_{5,5} := |\{e_1,\ e_2,\ e_3,\ e_4\}| = 4.$$

Forest matrix theorem

Theorem

For every graph G *and* $0 < k < |V|$,

The matrix $F[G, k]$ *satisfies* (Hyp).

This theorem implies Mason's graphic conjecture
(proof next slide).

Forest matrix theorem implies Mason's conjecture

Let

$$M \;:=\; F[G,k], \;\; \boldsymbol{x} \;:=\; (1,\ldots,1,0), \;\; \boldsymbol{y} \;:=\; (0,\ldots,0,1).$$

Then

$$\langle \boldsymbol{x}, M\boldsymbol{y}\rangle \;=\; k!\,a_k, \quad \langle \boldsymbol{x}, M\boldsymbol{x}\rangle \;=\; (k+1)!\,a_{k+1},$$
$$\langle \boldsymbol{y}, M\boldsymbol{y}\rangle \;=\; (k-1)!\,a_{k-1}.$$

$$\langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y}\rangle^2 \geq \; \langle \boldsymbol{x}, \boldsymbol{M}\boldsymbol{y}\rangle\,\langle \boldsymbol{y}, \boldsymbol{M}\boldsymbol{y}\rangle \quad (\text{Hyp})$$

then implies

$$a_k^2 \;\geq\; \left(1+\tfrac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

Forest matrix theorem

Theorem

For every graph G and $0 < k < |V|$,

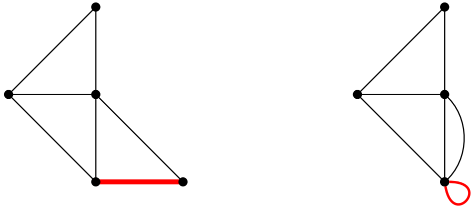
$$\text{The matrix } F[G,k] \text{ satisfies (Hyp).}$$

We will prove this theorem today by atlas method.

Forest atlas

Edge contraction

The contraction of edge e of G is the graph G/e where endpoints of e are merged to one.



G G/e

Loops and parallel edges are not removed.

In particular, there is a bijection between edges of G and G/e.

Forest atlas

Fix $t \in (0,1)$ and $d := m+1$ and $2 \leq k < |V|$.

Forest atlas $(M_0, \dots, M_d, \boldsymbol{h})$ is given by

$$
\begin{aligned}
M_0 &:= t\,F[G, k] \; + \; (1-t)\,F[G, k-1]; \\
M_i &:= F[G/e_i, k-1] \quad (i \in [m]); \\
M_d &:= F[G, k-1]; \\
\boldsymbol{h} &:= (t, \dots, t, 1-t).
\end{aligned}
$$

We will show that $M_0, \dots, M_d$ satisfy (Hyp), by showing that forest atlas satisfies some technical properties.

Technical properties of atlas that imply (Hyp)

Property 1–2: Irreducibility and h-Positivity

Atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ satisfies (Irr) if

matrix M_0 is irreducible.

Atlas satisfies (h-Pos) if

every entry of $\boldsymbol{h}$ is strictly positive.

Forest atlas satisfies (Irr) and (h-Pos)

W.l.o.g. G has no loops or parallel edges.

Then all non-diagonal entries of $F[G, k]$ are positive.

$$M_0 \;=\; t\,F[G, k] + (1 - t)F[G, k - 1]$$

is then irreducible, which implies (Irr).

$\boldsymbol{h} \;=\; (t, \ldots, t, 1{-}t)$ is strictly positive as $t \in (0, 1)$,

which implies (h-Pos). □

Property 3: Decreasing support

Atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ satisfies (Dec) if,

$$\operatorname{supp}(M_i) \subseteq \operatorname{supp}(M_0),$$

for every $i \in [d]$.

Here support of matrix M is

$$\operatorname{supp}(M) := \{ i \in [d] : M_{i,j} \neq 0 \text{ for some } j \in [d] \}.$$

Forest atlas satisfies (Dec)

W.l.o.g. G has no loops or parallel edges.

Then all non-diagonal entries of $F[G,k]$ are positive.

Then $\operatorname{supp}(F[G,k]) = [m+1] = [d]$.

Since

$$M_0 \;=\; t\,F[G,k] + (1-t)F[G,k-1],$$

It then follows that

$$\operatorname{supp}(M_i) \;\subseteq\; [d] = \operatorname{supp}(F[G,k]) \;=\; \operatorname{supp}(M_0),$$

which implies (Dec). □

Property 4: Transposition-Invariance

Atlas satisfies (T-Inv) if,

$$(M_i)_{j,\ell} \;=\; (M_j)_{\ell,i} \;=\; (M_\ell)_{i,j}$$

for every $i, j, \ell \in [d]$.

Here $(M_i)_{j,\ell}$ is the j, ℓ-th entry of matrix M_i.

Forest atlas satisfies (T-Inv)

W.l.o.g. $i, j, \ell \in [m]$. Then

$$(M_i)_{j,\ell} \;=\; (M_j)_{\ell,i} \;=\; (M_\ell)_{i,j}$$
$$=\; \Big|\{\text{forest words } e_i e_j e_\ell \underbrace{\cdots}_{k-2}\}\Big|,$$

and (T-Inv) follows. □

Property 5: Inheritance

Atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ satisfies (Inh) if,

$$i, j\text{-th entry of } M_0 \;=\; j\text{-th entry of } M_i \boldsymbol{h}\,,$$

for all $i, j \in [d]$.

Forest atlas satisfies (Inh)

W.l.o.g. $i, j \in [m]$. Then

i, j-th entry of M_0

$$= t(F[G,k])_{i,j} + (1-t)(F[G,k-1])_{i,j}$$

$$= t\left|\{\text{forest words } e_i \underbrace{\cdots}_{k-1} e_j \text{ of } G\}\right|$$

$$+ (1-t)\left|\{\text{forest words } e_i \underbrace{\cdots}_{k-2} e_j \text{ of } G\}\right|$$

$$= t\left|\{\text{forest words } \underbrace{\cdots}_{k-1} e_j \text{ of } G/e_i\}\right|$$

$$+ (1-t)\left|\{\text{forest words } \underbrace{\cdots}_{k-2} e_j \text{ of } G/e_i\}\right|$$

Forest atlas satisfies (Inh)

i, j-th entry of M_0

$$= \sum_{\ell=1}^{m} t \left| \{\text{forest words } e_\ell \underbrace{\cdots}_{k-2} e_j \text{ of } G/e_i\} \right|$$

$$+ (1-t) \left| \{\text{forest words } \underbrace{\cdots}_{k-2} e_j \text{ of } G/e_i\} \right|$$

$$= \sum_{\ell=1}^{m} h_\ell (F[G/e_i, k-1])_{\ell,j}$$

$$+ h_{m+1} (F[G/e_i, k-1])_{m+1,j}$$

$$= \langle \mathbf{1}_j, M_i \boldsymbol{h} \rangle = j\text{-th entry of } M_i \boldsymbol{h}. \qquad \square$$

Children-to-parent principle

Theorem (Theorem 3.4 (C.-Pak 22))

Let atlas $(M_0, \ldots, M_d, \boldsymbol{h})$ *satisfies* (Inh), (T-Inv), (Dec), (Irr), *and* (h-Pos). *Then*

$$M_1, \cdots, M_d \text{ satisfy (Hyp)} \implies M_0 \text{ satisfies (Hyp)}.$$

Thus our strategy becomes:

- Assume $M_1, \ldots, M_d$ satisfy (Hyp) (induction);
- Verify (Inh), (T-Inv), (Dec), (Irr), (h-Pos);
- $\implies$ M_0 satisfies (Hyp).

Proof of forest matrix theorem

Recall forest matrix theorem

Theorem

For every graph G and $0 < k < |V|$,

$$\text{The matrix } F[G,k] \text{ satisfies (Hyp).}$$

We will now prove this theorem (next slide).

Proof of forest matrix theorem

We will use induction on k. Base case is $k = 1$.

W.l.o.g. G has no loops or parallel edges. Then

$$F[G,1] = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

The eigenvalues are

$$\frac{(d-1)+\sqrt{(d-1)^2+4}}{2}, -1, \ldots, -1, \frac{(d-1)-\sqrt{(d-1)^2+4}}{2},$$

so $F[G,1]$ satisfies (OPE), and thus (Hyp).

Proof of forest matrix theorem

Assume $k \geq 2$.

Then $M_1, \ldots, M_d$ satisfy (Hyp) by induction.

Children-parent-principle $\implies$ M_0 satisfies (Hyp).

$\iff$ $tF[G,k] + (1-t)F[G,k-1]$ satisfies (Hyp).

$t \to 1$ $\implies$ $F[G,k]$ satisfies (Hyp). $\square$

What we have shown today

Let $G = (V, E)$ be a graph.

a_k = number of forests with k edges of graph G.

Forest is a subset of edges of G that has no cycles.

Theorem 1

For $0 < k < |V|$,

$$a_k^2 \geq \left(1 + \frac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

What we have shown today

Let $G = (V, E)$ be a graph.

a_k = number of forests with k edges of graph G.

Forest is a subset of edges of G that has no cycles.

Theorem 1

For $0 < k < |V|$,

$$a_k^2 \geq \left(1 + \frac{1}{k}\right) a_{k+1}\, a_{k-1}.$$

In fact, we can prove stronger inequalities for more general objects.

Mason's conjecture for matroids

Object: Matroids

Matroid $\mathcal{M} = (X, \mathcal{I})$ is ground set X with collection of independent sets $\mathcal{I} \subseteq 2^X$.

Graphical matroids

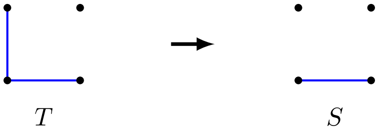
- X = edges of a graph G,
- $\mathcal{I}$ = forests in G.

Realizable matroids

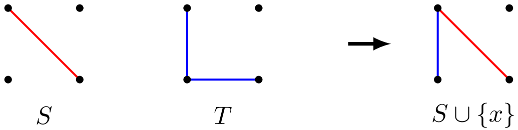
- X = finite set of vectors over field $\mathbb{F}$,
- $\mathcal{I}$ = sets of linearly independent vectors.

Matroids: Conditions

- $S \subseteq T$ and $T \in \mathcal{I}$ implies $S \in \mathcal{I}$.



- If $S, T \in \mathcal{I}$ and $|S| < |T|$, then there is $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.



Note: These are natural properties of sets of linearly independent vectors.

Mason's Conjecture (1972)

For every matroid and $k \geq 1$,

(1) $I_k{}^2 \;\geq\; I_{k+1}\, I_{k-1}$;

(2) $I_k{}^2 \;\geq\; \left(1 + \dfrac{1}{k}\right) I_{k+1}\, I_{k-1}$;

(3) $I_k{}^2 \;\geq\; \left(1 + \dfrac{1}{k}\right)\left(1 + \dfrac{1}{n-k}\right) I_{k+1}\, I_{k-1}$.

I_k is number of ind. sets of size k, and $n = |X|$.

Note: (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Why $\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right)$?

Mason (3) is equivalent to ultra/binomial log-concavity,

$$\frac{{I_k}^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}}{\binom{n}{k+1}} \frac{I_{k-1}}{\binom{n}{k-1}}.$$

Equality occurs **if** every $(k+1)$-subset is independent.

Solution to Mason (1)

Theorem (Adiprasito-Huh-Katz '18)

For every matroid and $k \geq 1$,

$$I_k^{\,2} \;\geq\; I_{k+1}\, I_{k-1}.$$

Proof used combinatorial Hodge theory for matroids.

Solution to Mason (2)

Theorem (Huh-Schröter-Wang '18)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1}.$$

Proof used combinatorial Hodge theory for correlation inequality on matroids.

Solution to Mason (3)

Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)

For every matroid and $k \geq 1$,

$$I_k{}^2 \ \geq \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

Proof used theory of strong log-concave polynomials / Lorentzian polynomials.

Solution to Mason (3)

Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

In fact, this inequality can be further improved, and we will see this improvement in the next class.

Next episode preview

Improvement for graphical matroids

Theorem (C.-Pak)

For graphical matroid of simple connected graph $G = (V, E)$, and $k = |V| - 2$,

$$(I_k)^2 \geq \frac{3}{2}\left(1 + \frac{1}{k}\right) I_{k+1}\, I_{k-1},$$

with equality if and only if G is cycle graph.

Numerically better than Mason (3), because

$$\frac{3}{2} \geq 1 + \frac{1}{n-k} = 1 + \frac{1}{|E| - |V| + 2}$$

for G that is not tree.

Comparison with Mason (3)

New bound gives

$$\frac{(I_k)^2}{I_{k+1}\, I_{k-1}} \quad \geq \quad \frac{3}{2} \qquad \text{when } \ |E| - |V| \to \infty,$$

Meanwhile, Mason (3) bound only gives

$$\frac{(I_k)^2}{I_{k+1}\, I_{k-1}} \quad \geq \quad 1 \qquad \text{when } \ |E| - |V| \to \infty.$$

New bound is better numerically and asymptotically.

This bound will be proved by atlas method in the next class.

SEE YOU NEXT CLASS!

References: www.arxiv.org/abs/2110.10740

www.arxiv.org/abs/2203.01533

Webpage: www.math.rutgers.edu/~sc2518/

Email: sweehong.chan@rutgers.edu

Method: Combinatorial atlas

Results: Log-concave inequalities, and
if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.

Method: Combinatorial atlas

Results: Log-concave inequalities, and
if and only if conditions for equality

- **Matroids (refined)**;
- Morphism of matroids (refined);
- Discrete polymatroids;
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.

Combinatorial atlas application: Matroids

Refinement for Mason (3)

Theorem 2 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1}\, I_{k-1}.$$

This refines Mason (3),

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k+1}\, I_{k-1},$$

since

$$\mathrm{prl}_{\mathcal{M}}(k-1) \;\leq\; n - k + 1.$$

Refinement for different matroids

- For all matroids,

  $${I_k}^2 \;\geq\; \left(1+\tfrac{1}{k}\right)\left(1+\tfrac{1}{n-k}\right) I_{k+1}\, I_{k-1}.$$

- Graphical matroids and $k = |V| - 2$,

  $${I_k}^2 \;\geq\; \left(1+\tfrac{1}{k}\right) \tfrac{3}{2}\, I_{k+1}\, I_{k-1}.$$

- Realizable matroids over $\mathbb{F}_q$,

  $${I_k}^2 \;\geq\; \left(1+\tfrac{1}{k}\right)\left(1+\tfrac{1}{q^{m-k+1}-2}\right) I_{k+1}\, I_{k-1}.$$

- (k, m, n)-Steiner system matroid,

  $${I_k}^2 \;\geq\; \left(1+\tfrac{1}{k}\right) \tfrac{n-k+1}{n-m}\, I_{k+1}\, I_{k-1}.$$

Refinement for Mason (3)

Theorem 3 (C.-Pak)

For every matroid and $k \geq 1$*,*

$$I_k{}^2 \ \geq \ \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1)-1}\right) I_{k+1}\, I_{k-1}.$$

This refines Mason (3),

$$I_k{}^2 \ \geq \ \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) I_{k+1}\, I_{k-1},$$

since

$$\mathrm{prl}_{\mathcal{M}}(k-1) \ \leq \ n-k+1.$$

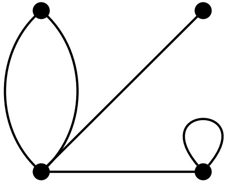
Parallel classes of matroid $\mathcal{M}$

Loop is $x \in X$ such that $\{x\} \notin \mathcal{I}$.

Non-loops x, y are parallel if $\{x, y\} \notin \mathcal{I}$.

Parallelship equiv. relation: $x \sim y$ if $\{x, y\} \notin \mathcal{I}$.

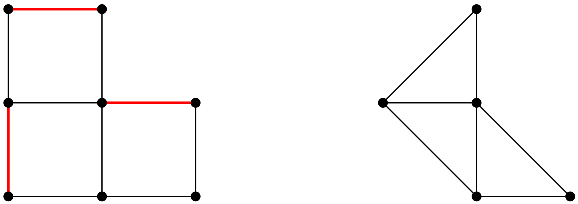
Parallel class $=$ equivalence class of $\sim$.



Matroid contraction

Contraction of $S \in \mathcal{I}$ is matroid $\mathcal{M}_S$ with

$$X_S = X \setminus S, \qquad \mathcal{I}_S = \{T \setminus S : S \subseteq T\}.$$

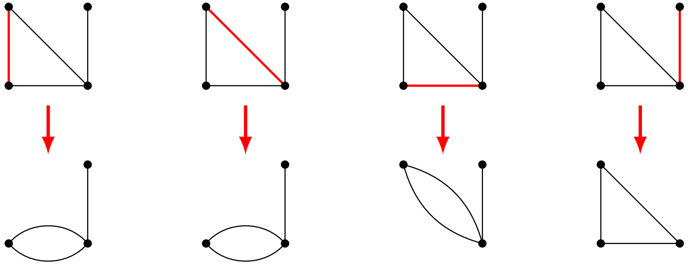


$\mathrm{prl}(S)$:= number of parallel classes of $\mathcal{M}_S$

Parallel number

The *k-parallel number* is

$$\mathrm{prl}_{\mathcal{M}}(k) \ := \ \max\{\mathrm{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$



$$\mathrm{prl}(S) = 2 \qquad \mathrm{prl}(S) = 2 \qquad \mathrm{prl}(S) = 2 \qquad \mathrm{prl}(S) = 3$$

$$\mathrm{prl}_{\mathcal{M}}(1) = 3$$

Refinement for Mason (3)

Theorem 4 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1}\, I_{k-1}.$$

This refines Mason (3),

$$I_k{}^2 \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n - k}\right) I_{k+1}\, I_{k-1},$$

since

$$\mathrm{prl}_{\mathcal{M}}(k-1) \;\leq\; n - k + 1.$$

When is equality achieved?

- When every $(k+1)$-subset is independent,
  $$\mathrm{prl}_{\mathcal{M}}(k-1) \ = \ n-k+1.$$
- Graphical matroid when G is a cycle,
  $$\mathrm{prl}_{\mathcal{M}}(k-1) \ = \ 3.$$
- Realizable matroids of every m-vectors over $\mathbb{F}_q$,
  $$\mathrm{prl}_{\mathcal{M}}(k-1) \ = \ q^{m-k+1}-1.$$
- (k,m,n)-Steiner system matroid,
  $$\mathrm{prl}_{\mathcal{M}}(k-1) \ = \ \frac{n-k+1}{m-k+1}.$$

Equality conditions

Theorem 5 (C.-Pak)

For every matroid and $k \geq 1$,

$$I_k{}^2 = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1}\, I_{k-1}$$

if and only if

for every $S \in \mathcal{I}$ *with* $|S| = k - 1$,

- $\mathcal{M}_S$ *has* $\mathrm{prl}_{\mathcal{M}}(k-1)$ *parallel classes; and*
- *Every parallel class of* $\mathcal{M}_S$ *has same size.*