

Four Functions Theorem and Schur Positivity

Swee Hong Chan

joint with Hong Chen, Igor Pak, and Daniel Soskin

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Fantasy World Cup 2026

Supercomputer predicts the probabilities for winning the final of Fantasy World Cup:



Argentina
39%



Spain
61%

Fantasy World Cup 2026

Suppose that **Messi suddenly got food poisoning.**

The supercomputer then updates the projected probabilities:



Argentina

32% ↓



Spain

68% ↑

Fantasy World Cup 2026



Argentina

32% ↓



Spain

68% ↑

The event: Messi got food poisoning.

- **Negative** correlation: Argentina's probability of winning World Cup **decreases**.
- **Positive** correlation: Spain's probability of winning World Cup **increases**.

Correlations: Definition

Events A and B are **positively correlated** if

$$P(A \cap B) \geq P(A)P(B).$$

If $P(B) > 0$, this is equivalent to

$$P(A \mid B) \geq P(A).$$

That is, knowing B makes A **more** likely.

Negative correlation is defined by **reversing**
the inequalities.

Example: negative correlations in spanning trees

Let G be a graph.

Let e, f be distinct edges of G .

Let T be a uniform random spanning tree of G .

Theorem (folklore Kirchhoff 1847)

$\{e \in T\}$ and $\{f \in T\}$ are *negatively correlated*,

$$P(e, f \in T) \leq P(e \in T)P(f \in T).$$

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$\{e \in T\}$ and $\{f \in T\}$ are *negatively correlated*,

$$P(e, f \in T) \leq P(e \in T)P(f \in T).$$

Intuition: Because T has a fixed number of edges, e and f are competing for the same limited spots.

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The inequality was proved via electrical networks;
modern proof uses determinantal processes.

Example: negative correlations in forests

Let G be a graph.

Let e, f be distinct edges of G .

Let F be a uniform random forest of G .

Question (Kahn '00, Grimmett–Winkler '04)

Are $\{e \in F\}$ and $\{f \in F\}$ are negatively correlated?

Forests satisfy the finite-slot intuition, and yet negative correlations remain an open problem.

Example: Erdős–Rényi random graphs

Consider **random graph** $G(n, p)$ with n vertices and independent edge probability $p = \frac{1}{2}$.

Let $A = \{G(n, p) \text{ is Hamiltonian}\}$.

Let $B = \{G(n, p) \text{ is planar}\}$.

Theorem

A and B are *negatively correlated*.

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Theorem

A and B are *negatively correlated*.

Intuition: Being Hamiltonian demands **more** edges, but being planar demands **fewer** edges.

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Let $A = \{G(n, p) \text{ is Hamiltonian}\}$.

Let $B = \{G(n, p) \text{ is planar}\}$.

Theorem

A and B are *negatively correlated*.

This result can be proved rigorously using the **Ahlsweide–Daykin inequality**.

**Ahlsvede–Daykin inequality
(Four Functions Theorem)**

Ahlsvede–Daykin (AD) inequality



Theorem (Ahlsvede–Daykin 1978)

Let $a, b, c, d : \mathbb{Z}^\ell \rightarrow \mathbb{R}_{\geq 0}$ be nonnegative functions satisfying

$$a(x)b(y) \leq c(x \vee y)d(x \wedge y) \quad \forall x, y \in \mathbb{Z}^\ell.$$

Then

$$\left(\sum_{x \in \mathbb{Z}^\ell} a(x) \right) \left(\sum_{x \in \mathbb{Z}^\ell} b(x) \right) \leq \left(\sum_{x \in \mathbb{Z}^\ell} c(x) \right) \left(\sum_{x \in \mathbb{Z}^\ell} d(x) \right).$$

Here $\vee$ is pointwise maximum and
 $\wedge$ is pointwise minimum.

Ahlsvede–Daykin (AD) inequality



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Intuition: **Local** log-supermodularity implies
global log-supermodularity.

Anecdotes: AD inequality

Quote (Ahlsvede)

"When Daykin visited Bielefeld, I was on a ladder wallpapering as Daykin explained a complicated new inequality. I instead proposed a simpler, more general version from the top of the ladder: the birth of the Ahlsvede–Daykin inequality."



Ahlsvede



Daykin

Anecdotes: AD inequality

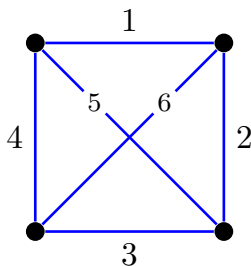
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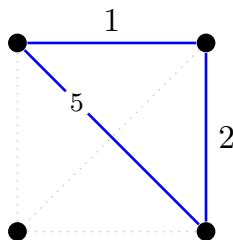


Proof: Hamiltonian-planar correlation

Each $x \in \{0, 1\}^{\binom{n}{2}}$ corresponds to a possible outcome of $G(n, p)$.



Base graph K_4



$x = (1, 1, 0, 0, 1, 0)$

Proof: Planar-Hamiltonian correlation

Let $a, b, c, d : \{0, 1\}^{\binom{n}{2}} \rightarrow \mathbb{R}$ be given by

$$a(x) := \mathbf{1}\{x \text{ is Hamiltonian and planar}\};$$

$$b(x) := 1$$

$$c(x) := \mathbf{1}\{x \text{ is Hamiltonian}\};$$

$$d(x) := \mathbf{1}\{x \text{ is planar}\}.$$

Proof: Hamiltonian-planar correlation

Being **Hamiltonian** is a **increasing** property:

x is Hamiltonian $\implies x \vee y$ is Hamiltonian.

Being **planar** is an **decreasing** property:

x is planar $\implies x \wedge y$ is planar.

This implies, for all $x, y \in \{0, 1\}^{\binom{n}{2}}$,

$$\underbrace{a(x)}_{\substack{\text{Hamiltonian,} \\ \text{planar}}} b(y) \leq \underbrace{c(x \vee y)}_{\text{Hamiltonian}} \underbrace{d(x \wedge y)}_{\text{planar}}.$$

Proof: Hamiltonian-planar correlation

$$\underbrace{a(x)}_{\substack{\text{Hamiltonian,} \\ \text{planar}}} b(y) \leq \underbrace{c(x \vee y)}_{\text{Hamiltonian}} \underbrace{d(x \wedge y)}_{\text{planar}}$$

is because:

- if $a(x)b(y) = 0$, then LHS is 0, and inequality follows.
- If $a(x)b(y) > 0$:
 - Then x is Hamiltonian and planar;
 - Then $x \vee y$ is Hamiltonian, and $x \wedge y$ is planar;
 - Then $a(x) = b(y) = c(x \vee y) = d(x \wedge y) = 1$, so inequality follows.

Proof: Hamiltonian-planar correlation

AD inequality thus gives us:

$$\underbrace{\left(\sum_x a(x)\right)}_{2^{\binom{n}{2}} P(A \cap B)} \underbrace{\left(\sum_x b(x)\right)}_{2^{\binom{n}{2}}} \leq \underbrace{\left(\sum_x c(x)\right)}_{2^{\binom{n}{2}} P(A)} \underbrace{\left(\sum_x d(x)\right)}_{2^{\binom{n}{2}} P(B)},$$

where

$A := \{G(n, p) \text{ is Hamiltonian}\}.$

$B := \{G(n, p) \text{ is planar}\}.$

Therefore A and B are **negatively correlated**.

Q.E.D.

AD inequality is a remarkably simple tool,
yet yields powerful results across
probability, order theory, combinatorics.

Today's focus: its applications to
Schur polynomials.

Schur polynomials

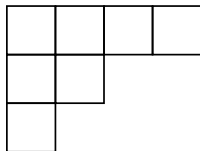
Integer partitions



Integer partition is a vector $\lambda \in \mathbb{Z}^\infty$ with finitely many nonzero entries, satisfying

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0.$$

Each partition visually corresponds to a Young diagram.



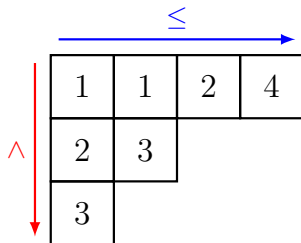
$$\lambda = (4, 2, 1, 0, \dots)$$

We denote by $\mathbb{Y}$ the set of all integer partitions.

Semi-standard Young tableau (SSYT)

An **SSYT** of shape λ and weight μ is a filling of λ where

- Rows are weakly increasing;
- Columns are strictly increasing;
- Each integer i appears exactly μ_i times.



$$\lambda = (4, 2, 1), \mu = (2, 2, 2, 1).$$

Schur polynomials

For a partition λ , Schur polynomial $s_\lambda := s_\lambda(\mathbf{x})$ is

$$s_\lambda := \sum_{T \in \text{SSYT}(\lambda)} x^{\mu(T)},$$

where $\mu(T)$ is the weight of the tableau T .

Schur polynomials

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- They are **characters** of irreducible representations of the general linear groups.
- They are **symmetric**, and form a **linear basis** for the space of all symmetric polynomials.

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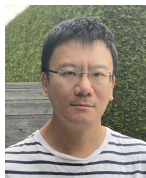
Today, we focus on the **correlation inequalities** exhibited by Schur polynomials.

Lam–Postnikov–Pylyavskyy inequality

LPP inequality (informal)

Theorem (Lam–Postnikov–Pylyavskyy 2007)

*Schur polynomial s_λ act as a **log-supermodular** function with respect to the partition λ .*



Lam



Postnikov



Pylyavskyy

LPP inequality



For symmetric polynomials f, g , we write

$f \leq_s g \iff$ Schur expansion of $g - f$
has nonnegative coefficients.

Theorem (Lam–Postnikov–Pylyavskyy 2007)

For any partitions λ and μ ,

$$s_\lambda s_\mu \leq_s s_{\lambda \vee \mu} s_{\lambda \wedge \mu},$$

where $\vee$ and $\wedge$ are defined coordinate-wise.

LPP inequality

Theorem (Lam–Postnikov–Pylyavskyy 2007)

For any partitions λ and μ ,

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where $\vee$ and $\wedge$ are defined coordinate-wise.

Okounkov (1997) conjectured this as a representation theory analogue of concavity of entropy functions.

LPP inequality

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For any partitions λ and μ ,

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Okounkov established **ordinary positivity**, leaving the more challenging **Schur positivity** as a conjecture.

LPP inequality

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For any partitions λ and μ ,

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where $\vee$ and $\wedge$ are defined coordinate-wise.

Inequality was established using Temperley–Lieb immanant and Kazhdan–Lusztig theory.

Searching for a simple proof

Quote (Atiyah)

“A proof by itself doesn't give you understanding. You can have a long proof and no idea at the end of why it works.”

Example of an elegant proof

Example: Cauchy–Davenport inequality

Theorem

Let p be prime. For any nonempty $A, B \subseteq \mathbb{Z}_p$,

$$|A + B| \geq \min\{p, |A| + |B| - 1\},$$

where $A + B$ is the sumset of A and B .

Proved by Cauchy (1813) and Davenport (1935)
using ideas from additive combinatorics.

A modern proof is given by
Combinatorial Nullstellensatz.

Combinatorial Nullstellensatz

Theorem (Alon 1999)

Theorem 1.2 *Let F be an arbitrary field, and let $f = f(x_1, \dots, x_n)$ be a polynomial in $F[x_1, \dots, x_n]$. Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then, if $S_1, \dots, S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$ so that*

$$f(s_1, \dots, s_n) \neq 0.$$



Alon

Proof of Cauchy–Davenport inequality

Proof of Theorem 3.2. If $|A| + |B| > p$ the result is trivial, since in this case for every $g \in Z_p$ the two sets A and $g - B$ intersect, implying that $A + B = Z_p$. Assume, therefore, that $|A| + |B| \leq p$ and suppose the result is false and $|A + B| \leq |A| + |B| - 2$. Let C be a subset of Z_p satisfying $A + B \subset C$ and $|C| = |A| + |B| - 2$. Define $f = f(x, y) = \prod_{c \in C} (x + y - c)$ and observe that by the definition of C

$$f(a, b) = 0 \text{ for all } a \in A, b \in B. \quad (5)$$

Put $t_1 = |A| - 1, t_2 = |B| - 1$ and note that the coefficient of $x^{t_1}y^{t_2}$ in f is the binomial coefficient $\binom{|A|+|B|-2}{|A|-1}$ which is nonzero in Z_p , since $|A| + |B| - 2 < p$. Therefore, by Theorem 1.2 (with $n = 2, S_1 = A, S_2 = B$), there is an $a \in A$ and a $b \in B$ so that $f(a, b) \neq 0$, contradicting (5) and completing the proof. $\square$

The elegant proof reveals the underlying combinatorial mechanism behind the inequality.

The discovery of Combinatorial Nullstellensatz led to a streamlined proof of Cauchy–Davenport inequality.

The discovery of **Combinatorial Nullstellensatz** led to a streamlined proof of **Cauchy–Davenport inequality**.

Is there a structural framework that leads to a streamlined proof of **LPP inequality**?

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Is there a structural framework that leads to a streamlined proof of **LPP inequality**?

We introduce **Ahlsvede–Daykin–Schur inequality** as a tool to prove **Schur positivity**, yielding a streamlined proof of **LPP inequality**.

Ahlsvede–Daykin–Schur inequality

Ahlsvede–Daykin–Schur inequality



Theorem (C.–Chen–Pak–Soskin 2026+)

Let $a, b, c, d : \mathbb{Y} \rightarrow \mathbb{R}_{\geq 0}$ be nonnegative functions satisfying

$$a(\alpha)b(\beta) \leq c(\alpha \vee \beta)d(\alpha \wedge \beta) \quad \forall \alpha, \beta \in \mathbb{Y}.$$

Then

$$\left(\sum_{\alpha \in \mathbb{Y}} a(\alpha) \right) \left(\sum_{\alpha \in \mathbb{Y}} b(\alpha) \right) \leq \left(\sum_{\alpha \in \mathbb{Y}} c(\alpha) \right) \left(\sum_{\alpha \in \mathbb{Y}} d(\alpha) \right).$$

Ahlsvede–Daykin–Schur inequality



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Then

$$\left(\sum_{\alpha \in \mathbb{Y}} a(\alpha) s_{\alpha} \right) \left(\sum_{\alpha \in \mathbb{Y}} b(\alpha) s_{\alpha} \right) \leq_s \left(\sum_{\alpha \in \mathbb{Y}} c(\alpha) s_{\alpha} \right) \left(\sum_{\alpha \in \mathbb{Y}} d(\alpha) s_{\alpha} \right).$$

Identical hypothesis as **classical AD inequality**, but
the conclusion now yields **Schur positivity**.

Ahlsvede–Daykin–Schur inequality



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We now apply this inequality to obtain a streamlined proof of **LPP inequality**.

Proof of LPP inequality

Let λ and μ be partitions in **LPP inequality**. Set

$$\begin{aligned}a(\alpha) &:= \mathbf{1}\{\alpha = \lambda\}; & b(\alpha) &:= \mathbf{1}\{\alpha = \mu\}; \\c(\alpha) &:= \mathbf{1}\{\alpha = \lambda \vee \mu\}; & d(\alpha) &:= \mathbf{1}\{\alpha = \lambda \wedge \mu\}.\end{aligned}$$

ADS inequality then implies

$$\underbrace{\left(\sum_{\alpha \in \mathbb{Y}} a(\alpha) s_{\alpha}\right)}_{s_{\lambda}} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} b(\alpha) s_{\alpha}\right)}_{s_{\mu}} \leq_s \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} c(\alpha) s_{\alpha}\right)}_{s_{\lambda \vee \mu}} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} d(\alpha) s_{\alpha}\right)}_{s_{\lambda \wedge \mu}}.$$

Q.E.D.

**ADS inequality application:
Mihalcea conjecture**

(Stable symmetric) Grothendieck polynomials

For partition λ , the Grothendieck polynomial is

$$G_{\lambda}(\mathbf{x}) := \sum_{\mu \in R(\lambda)} (-1)^{|\mu| - |\lambda|} g_{\mu/\lambda} s_{\mu}(\mathbf{x}),$$

where

- $R(\lambda)$ contains partitions μ satisfying

$$\lambda_i \leq \mu_i \leq \lambda_i + i - 1 \quad \text{for all } i.$$

- $g_{\mu/\lambda}$ is the number of tableaux of μ/λ that are
 - Strictly increasing in rows and columns;
 - Every entry in i -th row is restricted to $\{1, \dots, i-1\}$.

Grothendieck polynomial can be viewed as the **K-theory sibling** of Schur polynomial.

It is thus natural to expect Grothendieck polynomials to satisfy **log-supermodularity**, just like Schur polynomials.

Mihalcea conjecture

Let $\tilde{G}_\lambda(\mathbf{x}) := (-1)^{|\lambda|} G_\lambda(-\mathbf{x})$.

Theorem (C.–Chen–Pak–Soskin 2026+)

For any partitions λ and μ ,

$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_s \tilde{G}_{\lambda \vee \mu}(\mathbf{x}) \tilde{G}_{\lambda \wedge \mu}(\mathbf{x}).$$

This establishes **Schur log-supermodularity** for
Grothendieck polynomials.

Mihalcea conjecture

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$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_s \tilde{G}_{\lambda \vee \mu}(\mathbf{x}) \tilde{G}_{\lambda \wedge \mu}(\mathbf{x}).$$

We apply **ADS inequality** to give a streamlined proof of this conjecture.

Proof of Mihalcea conjecture

$$a(\alpha) := \mathbf{1}\{\alpha \in R(\lambda)\} g_{\alpha/\lambda};$$

$$b(\alpha) := \mathbf{1}\{\alpha \in R(\mu)\} g_{\alpha/\mu};$$

$$c(\alpha) := \mathbf{1}\{\alpha \in R(\lambda \vee \mu)\} g_{\alpha/(\lambda \vee \mu)};$$

$$d(\alpha) := \mathbf{1}\{\alpha \in R(\lambda \wedge \mu)\} g_{\alpha/(\lambda \wedge \mu)}.$$

ADS inequality then implies

$$\underbrace{\left(\sum_{\alpha \in \mathbb{Y}} a(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\lambda}(\mathbf{x})} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} b(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\mu}(\mathbf{x})} \leq_s \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} c(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\lambda \vee \mu}(\mathbf{x})} \underbrace{\left(\sum_{\alpha \in \mathbb{Y}} d(\alpha) s_{\alpha}\right)}_{\tilde{G}_{\lambda \wedge \mu}(\mathbf{x})}.$$

Q.E.D.

Other applications of ADS inequality

We obtain **Schur log-supermodularity** for

- Schur polynomials (LPP inequality);
- Grothendieck polynomials (Mihalcea conjecture);
- Dual Grothendieck polynomials;
- Factorial Schur polynomials;
- Hybrid Grothendieck polynomials (Guo–Kang–Liu follow-up).

THANK YOU!



Preprint: www.arxiv.org/abs/2606.06688

Webpage: www.math.rutgers.edu/~sc2518/

Thomas–Yong conjecture

For symmetric polynomials f, g , we write

$$f \leq_{\tilde{\mathbf{g}}} g \iff \begin{array}{l} \text{Expansion of } g - f \text{ in } \{\tilde{G}_\lambda(\mathbf{x})\} \\ \text{basis has nonnegative coefficients.} \end{array}$$

Conjecture (Thomas–Yong 2009)

For any partitions λ and μ ,

$$\tilde{G}_\lambda(\mathbf{x}) \tilde{G}_\mu(\mathbf{x}) \leq_{\tilde{\mathbf{g}}} \tilde{G}_{\lambda \vee \mu}(\mathbf{x}) \tilde{G}_{\lambda \wedge \mu}(\mathbf{x}),$$

Mihalcea conjecture is a special case, as
Grothendieck positivity implies Schur positivity.