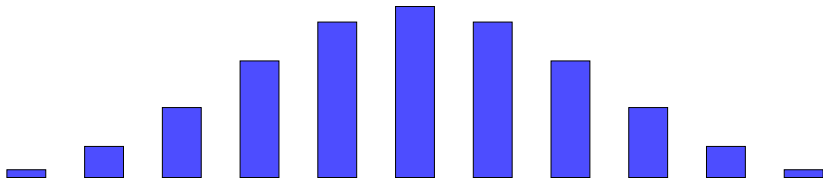


# Combinatorial Atlas for Log-concave Inequalities

Swee Hong Chan (UCLA)

joint with Igor Pak



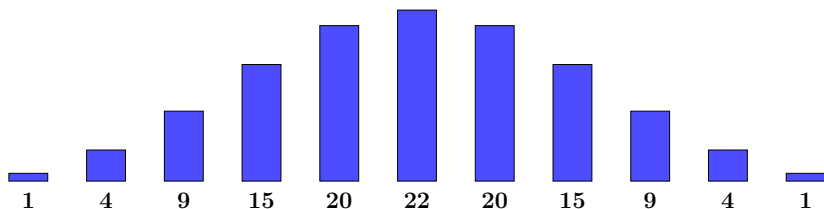
## What is log-concavity?

A sequence  $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$  is **log-concave** if

$$a_k^2 \geq a_{k+1} a_{k-1} \quad (1 \leq k < n).$$

Equivalently,

$$\log a_k \geq \frac{\log a_{k+1} + \log a_{k-1}}{2} \quad (1 \leq k < n).$$



## Example: binomial coefficients

$$a_k = \binom{n}{k} \quad k = 0, 1, \dots, n.$$

This sequence is **log-concave** because

$$\frac{a_k^2}{a_{k+1} a_{k-1}} = \frac{\binom{n}{k}^2}{\binom{n}{k+1} \binom{n}{k-1}} = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

The reasons of log-concavity for matroids was revealed in **June Huh's** lectures.

Today we focus on reasons from combinatorial perspective.

# Object: Matroids

Matroid  $\mathcal{M} = (X, \mathcal{I})$  is ground set  $X$  with collection of independent sets  $\mathcal{I} \subseteq 2^X$ .

## Graphical matroids

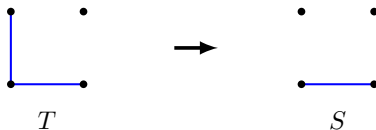
- $X$  = edges of a graph  $G$ ,
- $\mathcal{I}$  = forests in  $G$ .

## Realizable matroids

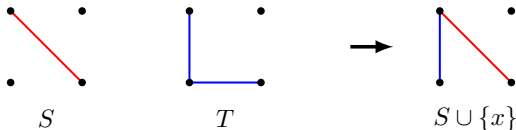
- $X$  = finite set of vectors over field  $\mathbb{F}$ ,
- $\mathcal{I}$  = sets of linearly independent vectors.

# Matroids: Conditions

- $S \subseteq T$  and  $T \in \mathcal{I}$  implies  $S \in \mathcal{I}$ .



- If  $S, T \in \mathcal{I}$  and  $|S| < |T|$ , then there is  $x \in T \setminus S$  such that  $S \cup \{x\} \in \mathcal{I}$ .



## Mason's Conjecture (1972)

For every matroid and  $k \geq 1$ ,

$$(1) \quad I_k^2 \geq I_{k+1} I_{k-1};$$

$$(2) \quad I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1};$$

$$(3) \quad I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$

$I_k$  is number of ind. sets of size  $k$ , and  $n = |X|$ .

Note: (3)  $\Rightarrow$  (2)  $\Rightarrow$  (1).

Why  $(1 + \frac{1}{k}) (1 + \frac{1}{n-k})$  ?

Mason (3) is equivalent to ultra/binomial log-concavity,

$$\frac{I_k^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}}{\binom{n}{k+1}} \frac{I_{k-1}}{\binom{n}{k-1}}.$$

Equality occurs **if** every subset with  $k + 1$  elements is independent.



## Solution to Mason (1)

### Theorem (Adiprasito-Huh-Katz '18)

*For every matroid and  $k \geq 1$ ,*

$$I_k^2 \geq I_{k+1} I_{k-1}.$$

Proof used [combinatorial Hodge theory](#) for  
matroids.

## Solution to Mason (2)

### Theorem (Huh-Schröter-Wang '18)

*For every matroid and  $k \geq 1$ ,*

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1}.$$

Proof used **combinatorial Hodge theory** for  
**correlation inequality** on matroids.

## Solution to Mason (3)

### Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)

*For every matroid and  $k \geq 1$ ,*

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$

Proof used theory of strong log-concave polynomials /  
Lorentzian polynomials.

## Solution to Mason (3)

### Theorem

(Anari-Liu-Oveis Gharan-Vinzant, Brändén-Huh '20)

*For every matroid and  $k \geq 1$ ,*

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$

### Theorem (Murai-Nagaoka-Yazawa '21)

*Equality occurs if and only if every subset with  $k + 1$  elements is independent.*

## **Our contribution**

**Method:** Combinatorial atlas

**Results:** Log-concave inequalities, and  
if and only if conditions for equality

- Matroids (refined);
- Morphism of matroids (refined);
- Discrete polymatroids (refined);
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.

**Method:** Combinatorial atlas

**Results:** Log-concave inequalities, and  
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- **Matroids (refined);**
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- Poset antimatroids;
- Branching greedoid (log-convex);
- Interval greedoids.

**Combinatorial atlas application:  
Matroids**



## Warmup: graphical matroids refinement

### Corollary (C.-Pak)

For graphical matroid of simple connected graph  $G = (V, E)$ , and  $k = |V| - 2$ ,

$$(I_k)^2 \geq \frac{3}{2} \left(1 + \frac{1}{k}\right) I_{k+1} I_{k-1},$$

with equality *if and only if*  $G$  is cycle graph.

Numerically *better* than Mason (3), because

$$\frac{3}{2} \geq 1 + \frac{1}{n-k} = 1 + \frac{1}{|E| - |V| + 2}$$

for  $G$  that is not tree.

## Comparison with Mason (3)

Our bound gives

$$\frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq \frac{3}{2} \quad \text{when } |E| - |V| \rightarrow \infty,$$

Meanwhile, Mason (3) bound only gives

$$\frac{(I_k)^2}{I_{k+1} I_{k-1}} \geq 1 \quad \text{when } |E| - |V| \rightarrow \infty.$$

Our bound is **better** numerically and asymptotically.

## Refinement for Mason (3)

### Theorem 1 (C.-Pak)

For every matroid and  $k \geq 1$ ,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1} I_{k-1}.$$

This refines Mason (3),

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n - k}\right) I_{k+1} I_{k-1},$$

since

$$\text{prl}_{\mathcal{M}}(k-1) \leq n - k + 1.$$

# Refinement for different matroids

- For all matroids,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{k+1} I_{k-1}.$$

- Graphical matroids and  $k = |V| - 2$ ,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \frac{3}{2} I_{k+1} I_{k-1}.$$

- Realizable matroids over  $\mathbb{F}_q$ ,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{q^{m-k+1}-2}\right) I_{k+1} I_{k-1}.$$

- $(k, m, n)$ -Steiner system matroid,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \frac{n-k+1}{n-m} I_{k+1} I_{k-1}.$$

## Refinement for Mason (3)

### Theorem 2 (C.-Pak)

For every matroid and  $k \geq 1$ ,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1} I_{k-1}.$$

This refines Mason (3),

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n - k}\right) I_{k+1} I_{k-1},$$

since

$$\text{prl}_{\mathcal{M}}(k-1) \leq n - k + 1.$$

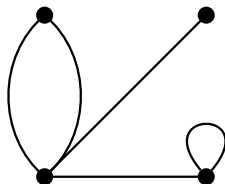
## Parallel classes of matroid $\mathcal{M}$

**Loop** is  $x \in X$  such that  $\{x\} \notin \mathcal{I}$ .

Non-loops  $x, y$  are **parallel** if  $\{x, y\} \notin \mathcal{I}$ .

**Parallelship** equiv. relation:  $x \sim y$  if  $\{x, y\} \notin \mathcal{I}$ .

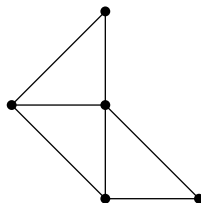
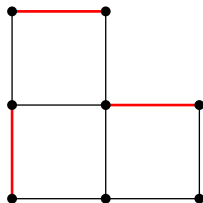
**Parallel class** = equivalence class of  $\sim$ .



# Matroid contraction

Contraction of  $S \in \mathcal{I}$  is matroid  $\mathcal{M}_S$  with

$$X_S = X \setminus S, \quad \mathcal{I}_S = \{T \setminus S : S \subseteq T\}.$$

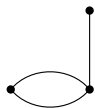


$\text{prl}(S) :=$  number of parallel classes of  $\mathcal{M}_S$

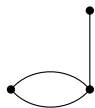
# Parallel number

The  $k$ -parallel number is

$$\text{prl}_{\mathcal{M}}(k) := \max\{\text{prl}(S) \mid S \in \mathcal{I} \text{ with } |S| = k\}.$$



$$\text{prl}(S) = 2$$



$$\text{prl}(S) = 2$$



$$\text{prl}(S) = 2$$



$$\text{prl}(S) = 3$$

$$\text{prl}_{\mathcal{M}}(1) = 3$$



## Refinement for Mason (3)

### Theorem 3 (C.-Pak)

For every matroid and  $k \geq 1$ ,

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1} I_{k-1}.$$

This refines Mason (3),

$$I_k^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n - k}\right) I_{k+1} I_{k-1},$$

since

$$\text{prl}_{\mathcal{M}}(k-1) \leq n - k + 1.$$

## When is equality achieved?

- When every  $(k + 1)$ -subset is independent,  
 $\text{prl}_{\mathcal{M}}(k - 1) = n - k + 1.$
- Graphical matroid when  $G$  is a cycle,  
 $\text{prl}_{\mathcal{M}}(k - 1) = 3.$
- Realizable matroids of every  $m$ -vectors over  $\mathbb{F}_q$ ,  
 $\text{prl}_{\mathcal{M}}(k - 1) = q^{m-k+1} - 1.$
- $(k, m, n)$ -Steiner system matroid,  
 $\text{prl}_{\mathcal{M}}(k - 1) = \frac{n - k + 1}{m - k + 1}.$

# Equality conditions

## Theorem 4 (C.-Pak)

For every matroid and  $k \geq 1$ ,

$$I_k^2 = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\text{prl}_{\mathcal{M}}(k-1) - 1}\right) I_{k+1} I_{k-1}$$

*if and only if*

for every  $S \in \mathcal{I}$  with  $|S| = k - 1$ ,

- $\mathcal{M}_S$  has  $\text{prl}_{\mathcal{M}}(k-1)$  parallel classes; and
- Every parallel class of  $\mathcal{M}_S$  has same size.

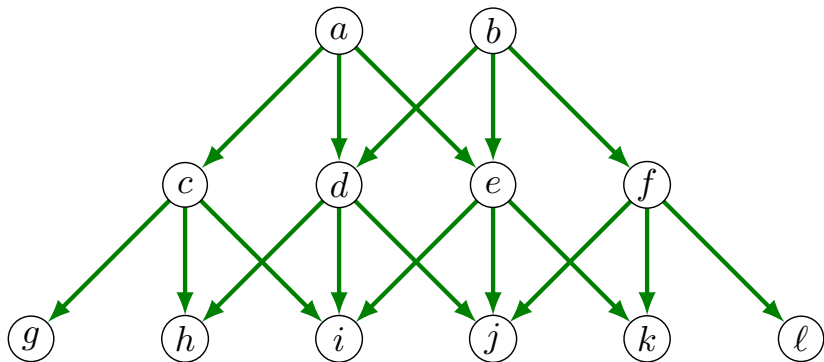
## **Combinatorial atlas: the method**

# Combinatorial atlas

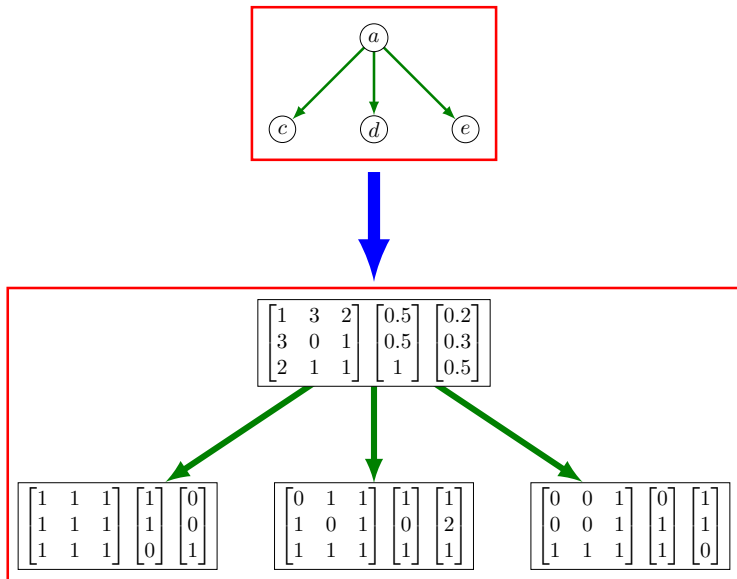
**Input:** Acyclic digraph  $\mathcal{A}$ , where each vertex  $v$  is associated with

- Symmetric matrix  $\mathbf{M}$  with nonnegative entries;
- Vector  $\mathbf{g}, \mathbf{h}$  with nonnegative entries.

## Atlas: example

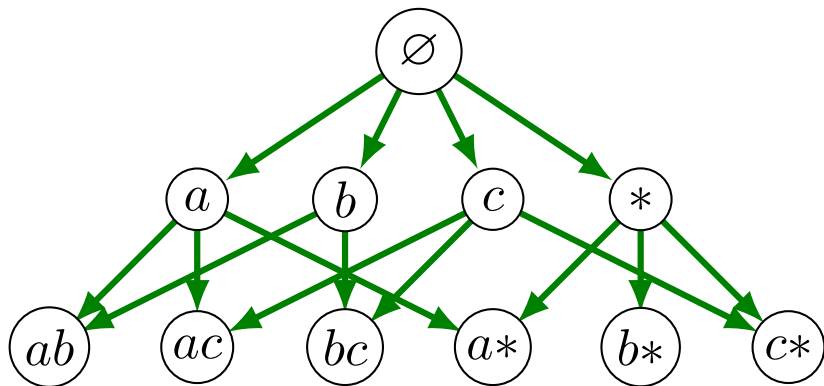


## Atlas: example (zoomed in)



## Atlas example: matroid (simplified)

For matroid with  $X = \{a, b, c\}$ , the atlas for  $k = 2$  is





## Atlas example: matroid (simplified)

The matrix for the top vertex is

$$M_{a,b} = (k + 1)! \times \text{number of independent sets} \\ \text{of size } k + 1 \text{ containing } a, b$$

$$M_{a,*} = k! \times \text{number of independent sets} \\ \text{of size } k \text{ containing } a$$

$$M_{*,*} = (k - 1)! \times \text{number of independent sets} \\ \text{of size } k - 1$$

# Combinatorial atlas

**Input:** Acyclic digraph  $\mathcal{A}$ , where each vertex  $v$  is associated with

- Symmetric matrix  $M$  with nonnegative entries;
- Vector  $g, h$  with nonnegative entries.

**Goal:** Show every  $M$  has hyperbolic inequality.

# Hyperbolic inequality

$M$  has hyperbolic inequality property if

$$\langle \mathbf{x}, M\mathbf{y} \rangle^2 \geq \langle \mathbf{x}, M\mathbf{x} \rangle \langle \mathbf{y}, M\mathbf{y} \rangle,$$

for every  $\mathbf{x} \in \mathbb{R}^r$ ,  $\mathbf{y} \in \mathbb{R}_{\geq 0}^r$ .

This condition is equivalent to

$M$  has at most one positive eigenvalue.

**Note:** Already known to be important in Lorentzian polynomials and Bochner's method proof of Aleksandrov-Fenchel inequality.

## How to get log-concave inequalities?

Assume  $a_{k-1}, a_k, a_{k+1}$  can be computed by

$$a_k = \langle \mathbf{g}, \mathbf{M}\mathbf{h} \rangle, \quad a_{k+1} = \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle, \quad a_{k-1} = \langle \mathbf{h}, \mathbf{M}\mathbf{h} \rangle,$$

for  $\mathbf{M}, \mathbf{g}, \mathbf{h}$  from a **top vertex** of the atlas.

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$$\langle \mathbf{g}, \mathbf{M}\mathbf{h} \rangle^2 \geq \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle \langle \mathbf{h}, \mathbf{M}\mathbf{h} \rangle \quad (\text{hyperbolic ineq.})$$

then implies

$$a_k^2 \geq a_{k+1} a_{k-1} \quad (\text{log-concave ineq.})$$

# Combinatorial atlas

**Input:** Acyclic digraph  $\mathcal{A}$ , where each vertex  $v$  is associated with

- Symmetric matrix  $M$  with nonnegative entries;
- Vector  $g, h$  with nonnegative entries.

**Goal:** Show every  $M$  has hyperbolic inequality.

**Method:** Verify three conditions:

- Irreducibility condition;
- Inheritance condition;
- Subdivergence condition.

# Irreducibility condition

- Matrix  $M$  associated to  $v$  is irreducible when restricted to its support;
  - Vector  $h$  is associated to  $v$  is a positive vector.
- 

For matroids, this means that the base exchange graph is connected.

This is a consequence of the exchange property.

## Inheritance condition

Edge  $e = (v, v_i)$  of  $v$  is associated with linear map  $T_i : \mathbb{R}^r \rightarrow \mathbb{R}^r$  such that, for every  $\mathbf{x} \in \mathbb{R}^r$ ,

$$i\text{-th coordinate of } \mathbf{M}\mathbf{x} = \langle T_i \mathbf{x}, \mathbf{M}_i T_i \mathbf{h} \rangle,$$

where  $\mathbf{M}$  and  $\mathbf{h}$  are associated to  $v$ , and  $\mathbf{M}_i$  is associated to  $v_i$ .

---

For **matroids** with  $X = \{e_1, \dots, e_n\}$ , this means

$$\begin{aligned} & k \times \text{number of independent } k\text{-sets} \\ &= \sum_{i=1}^n \text{number of independent } k\text{-sets containing } e_i. \end{aligned}$$

## Subdivergence condition

For every  $\mathbf{x} \in \mathbb{R}^r$ ,

$$\sum_{i=1}^r h_i \langle T_i \mathbf{x}, \mathbf{M}_i T_i \mathbf{x} \rangle \geq \langle \mathbf{x}, \mathbf{M} \mathbf{x} \rangle,$$

where  $h_i = i$ -th coordinate of  $\mathbf{h}$ .

**Note:** Equality occurs for Lorentzian polynomials  
and for matroids.

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For matroids, this is consequence of hereditary property.



# Bottom-to-top principle for hyperbolic inequalities

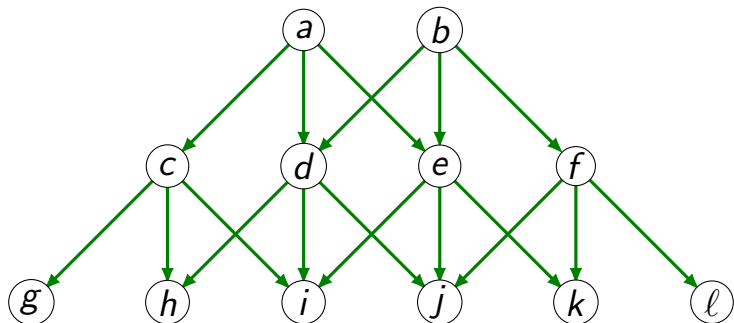
## Proposition

Assume *irreducibility, inheritance, subdivergence*.

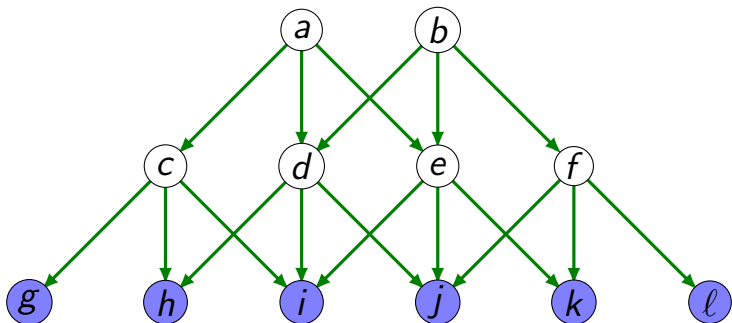
*If every child vertex has **hyperbolic inequality** property, then so does the parent vertex.*

Bottom-to-top principle **reduces Goal** to checking hyperbolic inequality only for **sink vertices**.

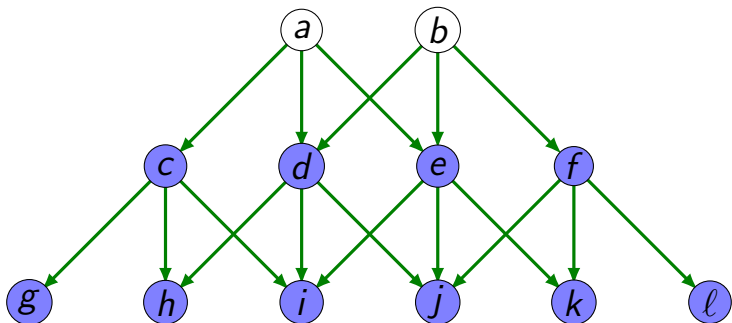
## Bottom-to-top principle



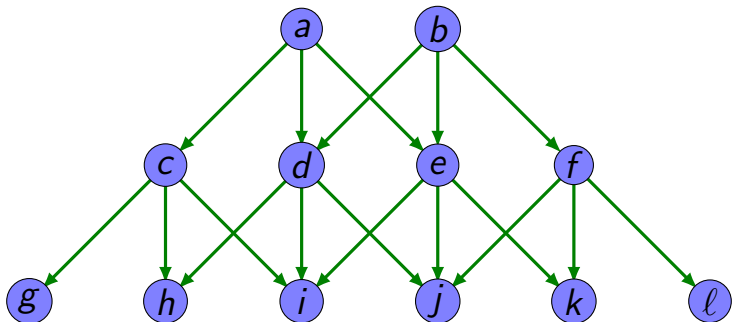
## Bottom-to-top principle



## Bottom-to-top principle



## Bottom-to-top principle



**How about equalities?**

# Combinatorial atlas equality

## Input:

- An atlas  $\mathcal{A}$  satisfying irreducibility, inheritance, subdivergence conditions.

**Goal:** Show “every”  $M$  has hyperbolic equality,

$$\langle g, Mh \rangle^2 = \langle g, Mg \rangle \langle h, Mh \rangle.$$

## Top-to-bottom principle for equalities

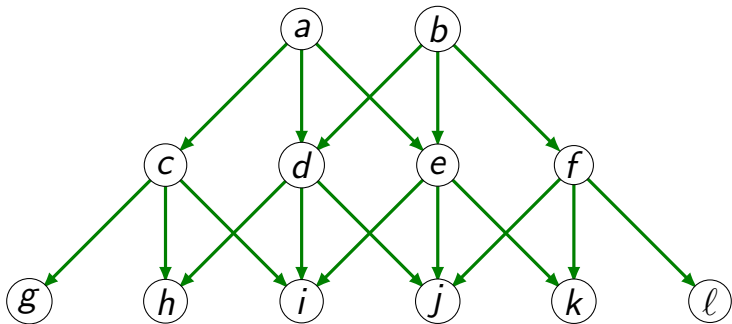
### Proposition

*If parent vertex has hyperbolic equality property, then so does children vertices.*

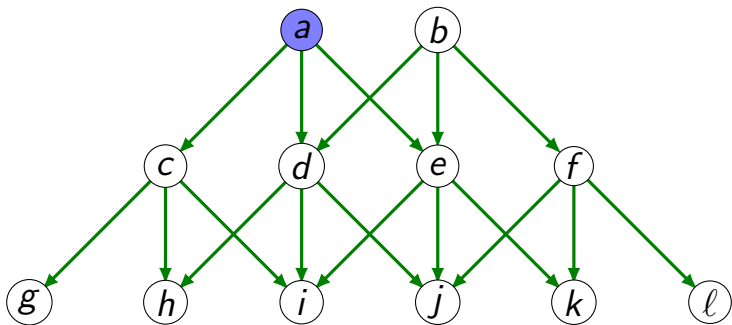
Top-to-bottom principle **expands** hyperbolic equality to sink vertices, and gives **combinatorial characterizations**.



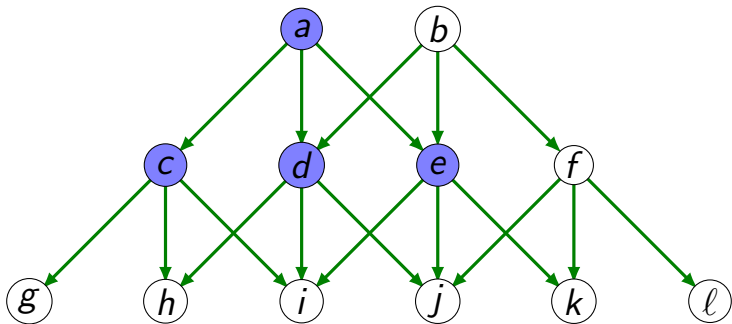
## Top-to-bottom principle



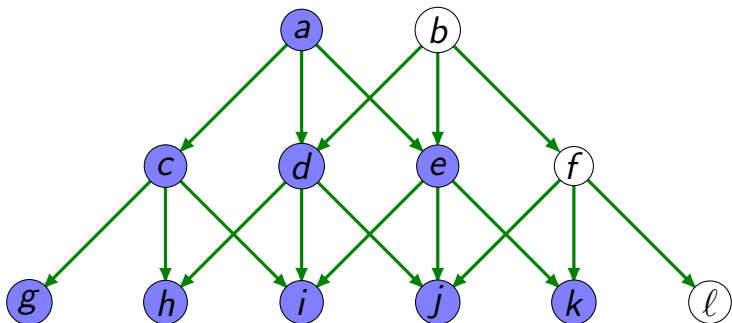
## Top-to-bottom principle



## Top-to-bottom principle



## Top-to-bottom principle



# Moral of the story

**Problem:** Log-concave **inequalities** and **equalities**.

**Strategy:**

- Build a **combinatorial atlas**;
- Verify the required conditions;
- Use **hyperbolic inequality** property to derive log-concave inequalities;
- Use **hyperbolic equality** property to derive log-concave equalities.

## Other applications

**Full version:** 2110.10740 (71 pages)

**Expository version:** 2203.01533 (28 pages)

**Results:** Log-concave inequalities and equalities for

- Matroids (refined);
- Discrete polymatroids (refined);
- Morphism of matroids (refined) ([conjecture on equality conditions is resolved](#));
- Stanley's poset inequality (refined);
- Poset antimatroids;
- Branching greedoid ([log-convex](#));
- Interval greedoids.

# THANK YOU!

Preprint: [www.arxiv.org/abs/2110.10740](http://www.arxiv.org/abs/2110.10740)

[www.arxiv.org/abs/2203.01533](http://www.arxiv.org/abs/2203.01533)

Webpage: [www.math.ucla.edu/~sweehong/](http://www.math.ucla.edu/~sweehong/)

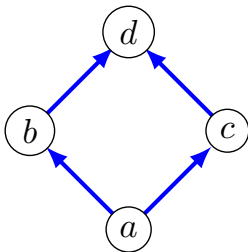
Email: [sweehong@math.ucla.edu](mailto:sweehong@math.ucla.edu)

**Combinatorial atlas application:  
Stanley's poset inequality**



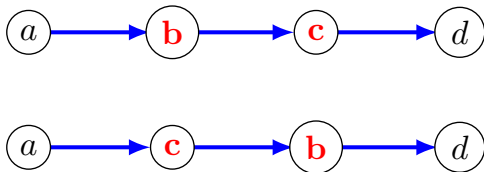
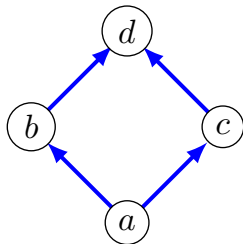
# Partially ordered sets

A poset  $P$  is a set  $X$  with a partial order  $\prec$  on  $X$ .



# Linear extension

A linear extension  $L$  is a complete order of  $\prec$ .



We write  $L(x) = k$  if  $x$  is  $k$ -th smallest in  $L$ .

## Stanley's inequality

Fix  $z \in P$ .

$N_k$  is number of linear extensions with  $L(z) = k$ .

### Theorem (Stanley '81)

For every poset and  $k \geq 1$ ,

$$N_k^2 \geq N_{k+1} N_{k-1}.$$

Proof used Aleksandrov-Fenchel inequality for mixed volumes.

## When is equality achieved?

### Theorem (Shenfeld-van Handel)

*Suppose  $N_k > 0$ . Then*

$$N_k^2 = N_{k+1} N_{k-1}$$

*if and only if*

$$N_k = N_{k+1} = N_{k-1}.$$

Proof used classifications of extremals of  
Aleksandrov-Fenchel inequality for convex polytopes.

# Our contribution

## Open Problem (Folklore)

Give a *combinatorial* proof to Stanley's inequality.

## Answer (C.–Pak)

We give new *combinatorial proof* for Stanley's ineq.  
and extend to *weighted version*.

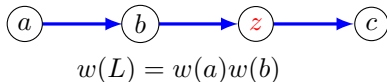
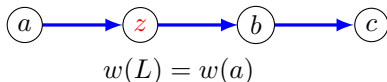
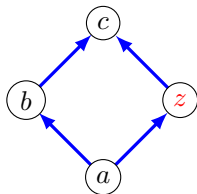
## Order-reversing weight

A weight  $w : X \rightarrow \mathbb{R}_{>0}$  is **order-reversing** if

$$w(x) \geq w(y) \quad \text{whenever} \quad x \prec y.$$

Weight of linear extension  $L$  is

$$w(L) := \prod_{L(x) < L(z)} w(x).$$



# Weighted Stanley's inequality

Fix  $z \in P$ .

$N_{w,k}$  is  $w$ -weight of linear extensions with  $L(z) = k$ .

## Theorem 5 (C. Pak)

For every poset and  $k \geq 1$ ,

$$N_{w,k}^2 \geq N_{w,k+1} N_{w,k-1}.$$

## When is equality achieved?

### Theorem 6 (C.-Pak)

Suppose  $N_{w,k} > 0$ . Then

$$N_{w,k}^2 = N_{w,k+1} N_{w,k-1}$$

*if and only if*

for every linear extension  $L$  with  $L(z) = k$ ,

$$w(L^{-1}(k+1)) = w(L^{-1}(k-1)) =: s,$$

and

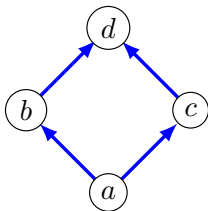
$$\frac{N_{w,k}}{s^k} = \frac{N_{w,k+1}}{s^{k+1}} = \frac{N_{w,k-1}}{s^{k-1}}.$$



**Combinatorial atlas application:  
Poset antimatroids**

## Feasible words of a poset

A word  $\alpha \in X^*$  is **feasible** if no repeating elements, and  $y$  occurs in  $\alpha$  and  $x \prec y \Rightarrow x$  occurs in  $\alpha$  before  $y$ .



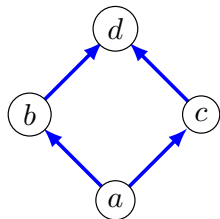
Feasible:  $\emptyset$ ,  $a$ ,  $ab$ ,  $ac$ ,  $abc$ ,  $acb$ ,  $abcd$ ,  $acbd$ .

Not feasible:  $aa$ ,  $bc$ ,  $ba$ .

## Chain weight

For  $x \in P$ , chain weight is

$\omega(x)$  = number of maximal chains that starts with  $x$ .

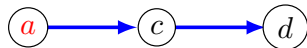
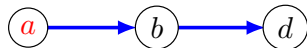


$$\omega(a) = 2$$

$$\omega(b) = 1$$

$$\omega(c) = 1$$

$$\omega(d) = 1$$



Weight of word  $\alpha$  is  $\omega(\alpha) := \omega(\alpha_1) \dots \omega(\alpha_\ell)$ .

# Log-concave inequality for poset antimatroids

$F_{\omega,k}$  is sum of  $\omega$ -weight of feasible words of length  $k$ .

## Theorem 7 (C.-Pak)

For every poset and  $k \geq 1$ ,

$$F_{\omega,k}^2 \geq F_{\omega,k+1} F_{\omega,k-1}.$$

## When is equality achieved?

### Theorem 8 (C.-Pak)

*Equality occurs for  $k = 1, \dots, \text{height}(P) - 1$   
if and only if*

*Hasse diagram of  $P$  is a forest where every leaf is of  
the same level.*

