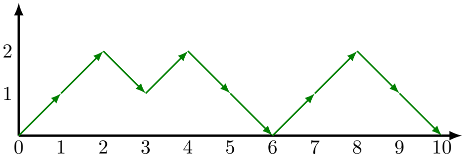
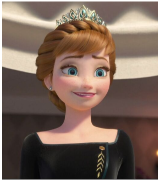
Sorting probability for Young diagrams

Swee Hong Chan (UCLA)

joint with Igor Pak and Greta Panova

1	2	4	7	8
3	5	6	9	10





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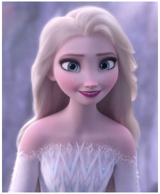

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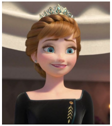
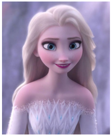
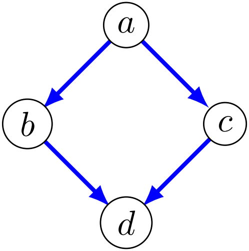
 $<$  $<$  $<$  $<$  $<$ 

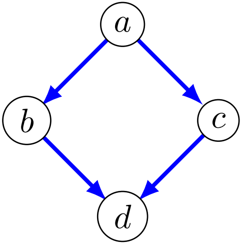
Partially ordered set

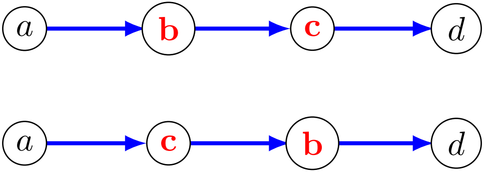
A poset P is a set X with a partial order $\preccurlyeq$ on X.



Linear extension

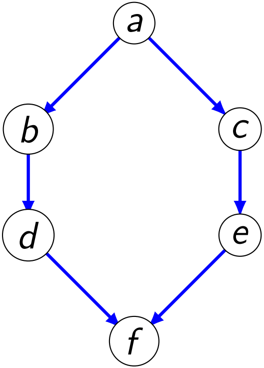
A linear extension L is a complete order of $\preccurlyeq$.





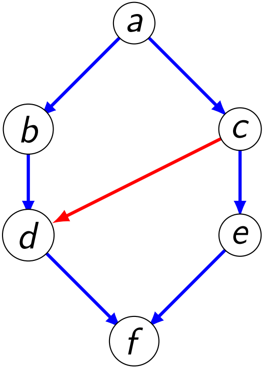
We write $e(P)$ for number of linear extensions of P.

How many steps needed to complete a partial order?



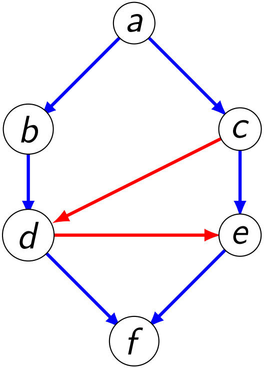
How many steps needed to complete a partial order?

We first compare c and d, and get $c \preccurlyeq d$.



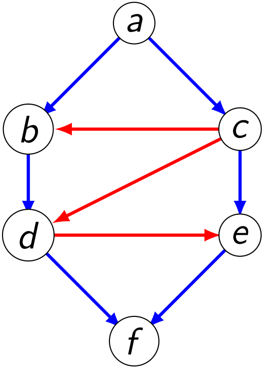
How many steps needed to complete a partial order?

We then compare d and e, and get $d \preccurlyeq e$.



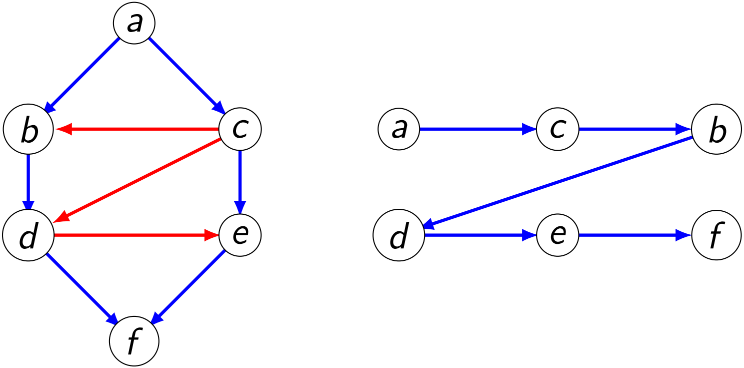
How many steps needed to complete a partial order?

We continue with b and c, and get $c \preccurlyeq b$.



How many steps needed to complete a partial order?

Completing the partial order took 3 steps.



Strategy to complete the partial order

At each step, compare x and y that satisfies

$$\frac{1}{2} - c \quad \leq \quad \mathrm{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{1}{2} + c\,,$$

where P is uniform on linear extensions of P.

Runtime is $\Theta(\log e(P))$ steps.

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For every finite poset that is not completely ordered, there exists x, y:

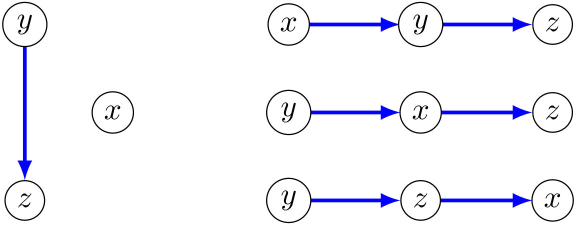
$$\frac{1}{3} \quad \leq \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{2}{3}.$$

(Brightwell-Felsner-Trotter '95)

"This problem remains one of the most intriguing problems in the combinatorial theory of posets."

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper,lower bound are achieved by this poset:



$$\mathsf{P}\left[x \preccurlyeq y\right] = \frac{1}{3}; \qquad \mathsf{P}\left[y \preccurlyeq x\right] = \frac{2}{3}.$$

What is known so far

Theorem (Kahn-Saks '84)

For every finite poset, there always exists x, y:

$$\frac{3}{11} \quad \le \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \le \quad \frac{8}{11},$$

roughly between 0.273 *and* 0.727.

Proof is by applying mixed-volume inequalities to order polytopes.

What is known so far

Theorem (Brightwell-Felsner-Trotter '95)

For every finite poset, there always exists x, y:

$$\frac{5-\sqrt{5}}{10} \quad \leq \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{5+\sqrt{5}}{10},$$

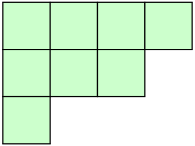
roughly between 0.276 *and* 0.724*.*

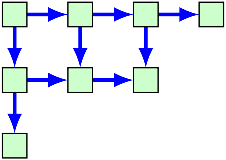
This bound cannot be improved for infinite posets.

Young diagrams

Elements of P_λ are cells of Young diagram of shape λ.

$x \preccurlyeq y \quad$ if y lies to the Southeast of x.





Young diagram of shape $\lambda = (4, 3, 1)$

We write n for number of cells of Young diagram.

Young diagrams

Linear extensions of P_λ correspond to standard Young tableau of the Young diagram.

1	2	5	6
3	4	7	
8			

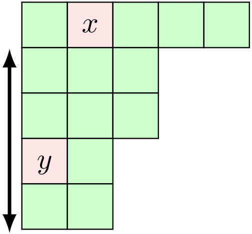
Linear extensions are counted by hook-length formulas.

What is known for Young diagrams

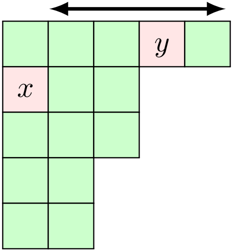
Theorem 1 (Olson–Sagan '18)

For Young diagrams, there always exists x, y:

$$\frac{1}{3} \quad \leq \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{2}{3}\,.$$



or



What is known for Young diagrams

Theorem 1 (Olson–Sagan '18)

For Young diagrams, there always exists x, y:

$$\frac{1}{3} \quad \leq \quad \mathrm{P}\big[x \preccurlyeq y\big] \quad \leq \quad \frac{2}{3}\,.$$

We sketch an alternative proof for Young diagrams using Naruse hook-length formulas.

Hook-length formulas

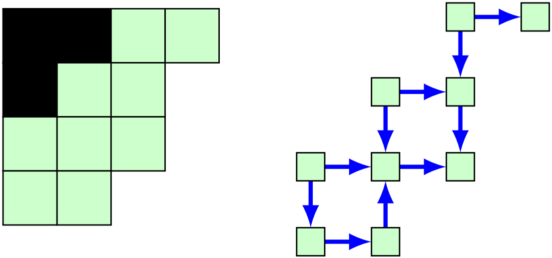
Number of standard Young tableau of shape λ is

$$f^{\lambda} \;:=\; \frac{n!}{\prod_{x\in\lambda} h_{\lambda}(x)}\,.$$

7	6	4	1
5	4	2	
4	3	1	
2	1		

$$f^{\lambda} \;=\; \frac{12!}{7\,6\,4\,1\,5\,4\,2\,4\,3\,1\,2\,1} \;=\; 2970$$

Skew Young diagrams



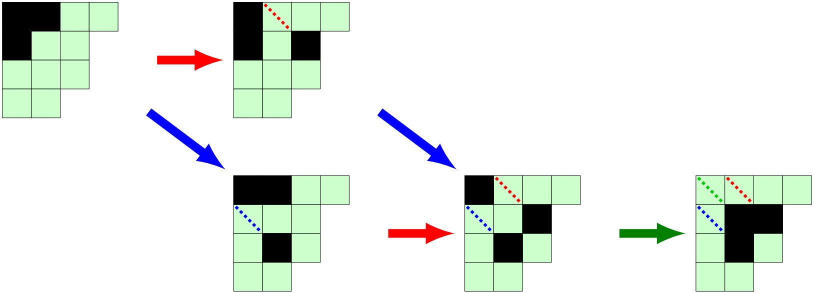
Skew Young diagram of shape λ/μ,
$\lambda = (5,3,3,1)$ and $\mu = (2,1)$.

We write *n* for number of cells in λ,
and *m* for number of cells in μ.

Excited diagrams

At each step, move a black box on SouthEast direction

- Boxes cannot leave the green diagram,
- Boxes cannot move if blocked by other boxes.



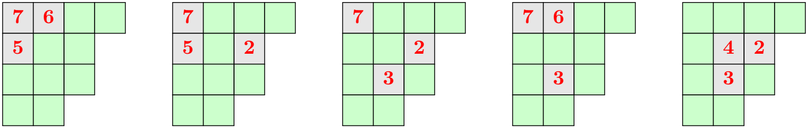
Naruse hook-length formulas

Theorem (Naruse '14, Morales-Pak-Panova '17)

Number of skew Young tableau of shape λ/μ is

$$f^{\lambda/\mu} \ := \ f^{\lambda}\frac{(n-m)!}{n!} \sum_{\substack{\text{excited} \\ \text{diagrams } B}} \prod_{\substack{\text{black cells} \\ x \in B}} h_{\lambda}(x)\,.$$

Naruse hook-length formulas



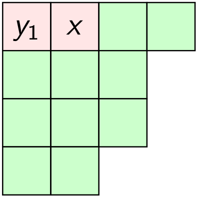
The number of SYT of shape λ/μ is equal to

$$2970 \, \frac{9!}{12!} \left(7 \cdot 6 \cdot 5 + 7 \cdot 5 \cdot 2 + 7 \cdot 2 \cdot 3 + 7 \cdot 6 \cdot 3 \; + \; 4 \cdot 2 \cdot 3 \right)$$

$$= \; 1062 \, .$$

Note: Every term in NHLF is nonnegative.

Proof of Theorem Olson–Sagan



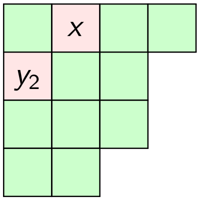
$$\mathrm{P}\big[\, x \preccurlyeq y_1 \,\big] \;=$$

The jump probabilities are

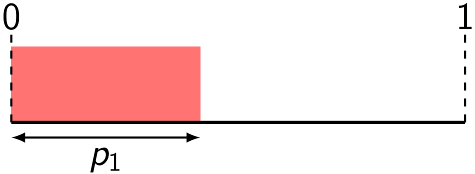
$$p_i \;:=\; \mathrm{P}\big[\, y_i \preccurlyeq x \preccurlyeq y_{i+1} \,\big],$$

where y_i is the i-th element in 1st column.

Proof of Theorem Olson–Sagan



$$P\left[x \preccurlyeq y_2\right] \;=$$



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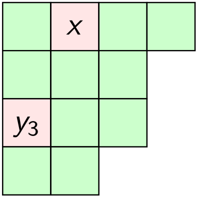
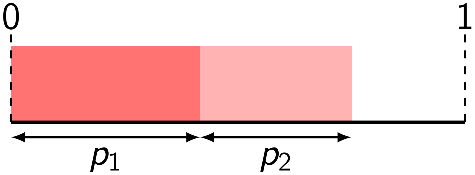


$$P\left[x \preccurlyeq y_3\right] \;=$$



The jump probabilities are

$$p_i \;:=\; P\left[y_i \preccurlyeq x \preccurlyeq y_{i+1}\right],$$

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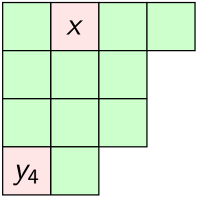
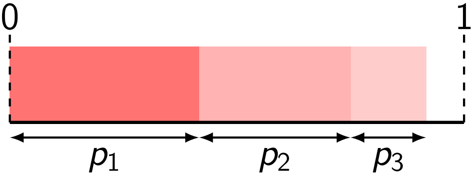
Proof of Theorem Olson–Sagan



$$\mathrm{P}\big[\, x \preccurlyeq y_4 \,\big] \;=$$



The jump probabilities are

$$p_i \;:=\; \mathrm{P}\big[\, y_i \preccurlyeq x \preccurlyeq y_{i+1} \,\big],$$

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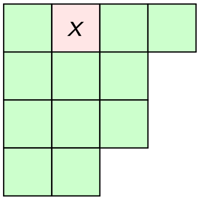
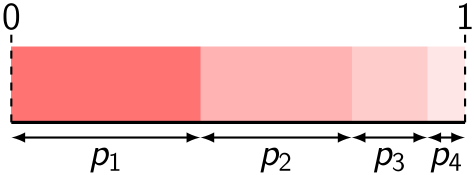
Proof of Theorem Olson–Sagan



$$\mathrm{P}\big[\, x \preccurlyeq y_5 \,\big] \; = $$



The jump probabilities are

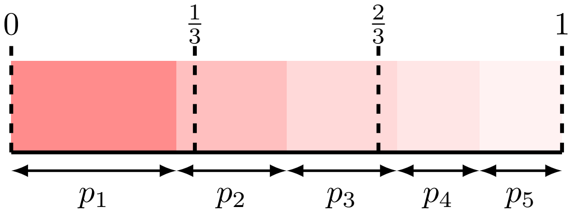
$$p_i \;:=\; \mathrm{P}\big[\, y_i \preccurlyeq x \preccurlyeq y_{i+1} \,\big],$$

where y_i is the i-th element in 1st column.

Linial-type argument

Supppose that $p_1, p_2, p_3, \ldots$ are all $< \frac{1}{3}$.

$\mathbf{P}\left[\, x \preccurlyeq y_i \,\right] \;=$



Look at when the probability exceeds $\frac{1}{3}$. Then

$$\frac{1}{3} \quad \leq \quad \mathsf{P}\left[x \preccurlyeq y_{i+1}\right] \quad \leq \quad \frac{2}{3}\,.$$

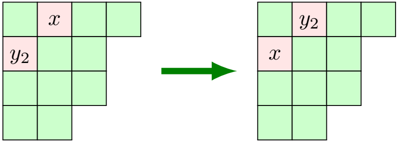
Proof of $p_1 < \frac{1}{3}$

Suppose to the contrary that $p_1 \geq \frac{1}{3}$. Then

- If $\frac{1}{3} \leq p_1 \leq \frac{2}{3}$, then

$$\frac{1}{3} \quad \leq \quad p_1 \;=\; \mathrm{P}\big[x \preccurlyeq y_2\big] \quad \leq \quad \frac{2}{3}.$$

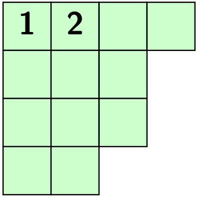
- If $p_1 > \frac{2}{3}$, then substitute $x \leftrightarrow y_2$ so $p_1 < \frac{1}{3}$.



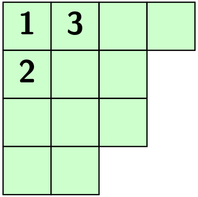
Skew diagrams enter the scene

It suffices to show $p_1 \geq p_2 \geq p_3 \geq \ldots$.

$$p_1 = \mathrm{P}\big[\, y_1 \preccurlyeq x \preccurlyeq y_2 \,\big] = \frac{\#\text{ of SYTs of}}{f^\lambda}$$



$$p_2 = \mathrm{P}\big[\, y_2 \preccurlyeq x \preccurlyeq y_3 \,\big] = \frac{\#\text{ of SYTs of}}{f^\lambda}$$



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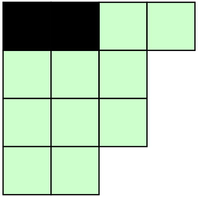

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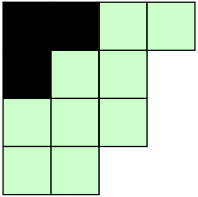
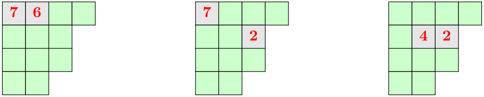


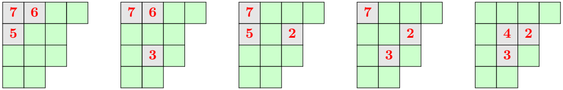
We can now use **NHLF**.

Proof of $p_1 \geq p_2$



(10!)(7)(6) (10!)(7)(2) (10!)(4)(2)

$$p_1 = \frac{\left(10! \cdot 7 \cdot 6 + 10! \cdot 7 \cdot 2 + 10! \cdot 4 \cdot 2\right)}{12!} = \frac{9!}{12!}\, 640\,.$$



(9!)(7)(6)(5) (9!)(7)(6)(3) (9!)(7)(2)(5) (9!)(7)(2)(3) (9!)(4)(2)(3)

$$p_2 = \frac{\left(9! \cdot 7 \cdot 6 \cdot 8 + 9! \cdot 7 \cdot 2 \cdot 8 + 9! \cdot 4 \cdot 2 \cdot 3\right)}{12!} = \frac{9!}{12!}\, 472\,.$$

Thus we complete the proof of this theorem.

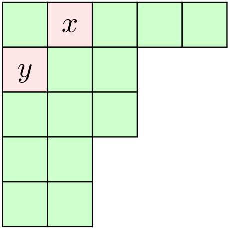
Theorem (Olson-Sagan '18)

There always exists x, y:

$$\frac{1}{3} \quad \leq \quad \mathrm{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{2}{3},$$

for poset P_λ of Young diagram of shape λ.

Back to previous example



Comparison probability for this Young diagram is

$$\mathsf{P}\big[\, x \preccurlyeq y \,\big] \quad = \quad \frac{16}{33} \quad \approx \quad 0.4848,$$

which is closer to $\frac{1}{2}$ than $\frac{1}{3}$, $\frac{2}{3}$.

What we will do next

Previously, we want to find x, y:

$$\frac{1}{3} \quad \leq \quad \mathsf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{2}{3}\,,$$

Now, we want to find x, y:

$$\frac{1}{2} \, - \, \delta \quad \leq \quad \mathsf{P}\big[\, x \preccurlyeq y \,\big] \quad \leq \quad \frac{1}{2} \, + \, \delta\,,$$

Sorting probability

Sorting probability of a poset P is

$$\delta(P) \ := \ \min_{\text{distinct } x, y} \Big| \mathrm{P}\,[x \prec y] - \mathrm{P}\,[y \prec x] \Big|.$$

In particular, there exists x, y:

$$\frac{1}{2} - \frac{\delta(P)}{2} \quad \leq \quad \mathrm{P}\Big[\, x \preccurlyeq y \,\Big] \quad \leq \quad \frac{1}{2} + \frac{\delta(P)}{2}.$$

Kahn–Saks Conjecture

Conjecture (Kahn-Saks '84)

For every finite poset,

$$\delta(P) \to 0 \quad \textit{as} \quad \textit{width}(P) \to \infty.$$

Here width(P) is the largest size of anti-chains in P.

Komlós '90 proved such a result for posets with $\Omega(\frac{n}{\log\log\log n})$ minimal elements.

Our results

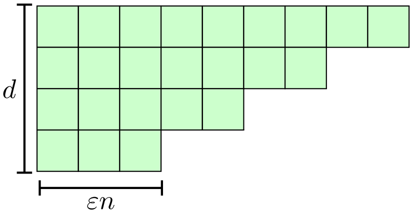
First result

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of Young diagram of λ,

$$\delta(P_\lambda) \ \leq \ \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.



First result

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of Young diagram of λ,

$$\delta(P_\lambda) \;\leq\; \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.

Proof ingredient:

NHLF + Random walk intuition

Where is the improvement?

Before: x is 2nd element in 1st row, y is in 1st column.

Intuition: Probability of SRW on $\mathbb{Z}$ to visit 0
at 2nd step is of constant order.

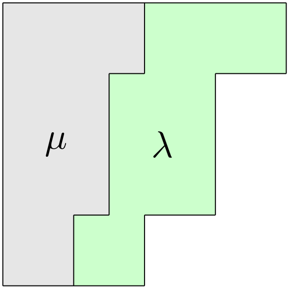
Now: x is midpoint of 1st row, y is in 2nd row.

Intuition: Probability of SRW on $\mathbb{Z}$ to visit 0
at $\frac{n}{2}$-th step is of the order of $\frac{1}{\sqrt{n}}$.

Sketch of proof

After reductions using Hoeffding's inequality,

$$\delta(P_\lambda) \;\le\; \sum_{\mu} \frac{\text{SYTs of}}{f^\lambda}$$



$$\text{with } \mu \;\approx\; \left(\frac{\lambda_1}{2} \pm \sqrt{n}, \ldots, \frac{\lambda_d}{2} \pm \sqrt{n}\right).$$

Right side is then upper-bounded via NHLF.

Back to first result

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of Young diagram of λ,

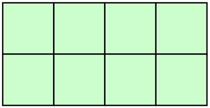
$$\delta(P_\lambda) \;\leq\; \frac{C}{\sqrt{n}},$$

for some $C = C(d, \varepsilon) > 0$.

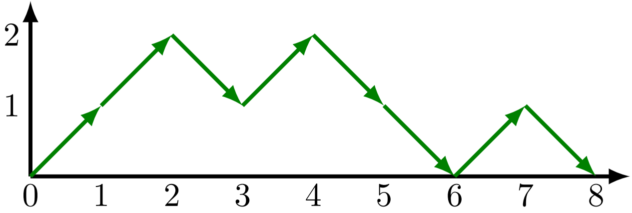
Next: better bound for Catalan posets.

Catalan posets, $\lambda = \left(\frac{n}{2}, \frac{n}{2}\right)$

Young diagram is rectangle with 2 rows and n cells.



1	2	4	7
3	5	6	8



Second result

Theorem (C.-Pak-Panova '21)

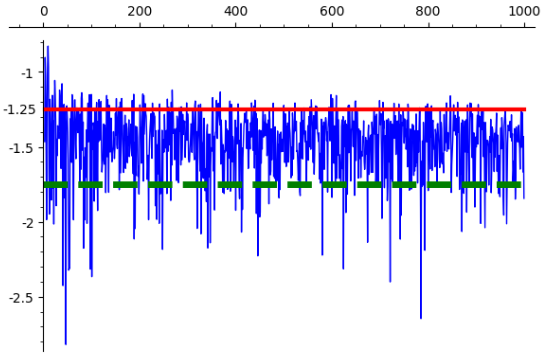
For Catalan posets with n cells,

$$\delta(P_\lambda) \;\le\; C n^{-\frac{5}{4}},$$

for some $C > 0$.

How good is this bound?

$$\frac{\log \delta(P_\lambda)}{n}$$



Open Problem

Show that

$$\limsup_{n\to\infty} \frac{\log \delta(P_\lambda)}{n} = -\frac{5}{4}; \quad \liminf_{n\to\infty} \frac{\log \delta(P_\lambda)}{n} < -\frac{5}{4}.$$

Where is the improvement?

For each x in 1st row, find $y(x)$ in 2nd row minimizing

$$\delta(x, y(x)) \; := \; \Big| \mathrm{P}\left[x \prec y(x)\right] - \mathrm{P}\left[y(x) \prec x\right] \Big| .$$

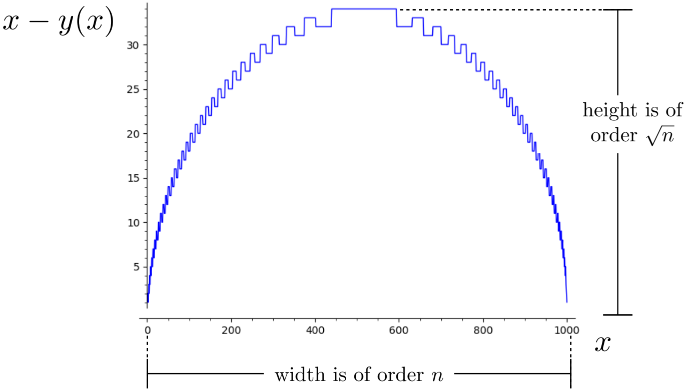
Before: x is fixed at midpoint of 1st row,

$$\delta(P_\lambda) \; \leq \; \delta(x, y(x)).$$

Now: Optimize over all x's in 1st row,

$$\delta(P_\lambda) \; \leq \; \min_{x \text{ in 1st row}} \; \delta(x, y(x)).$$

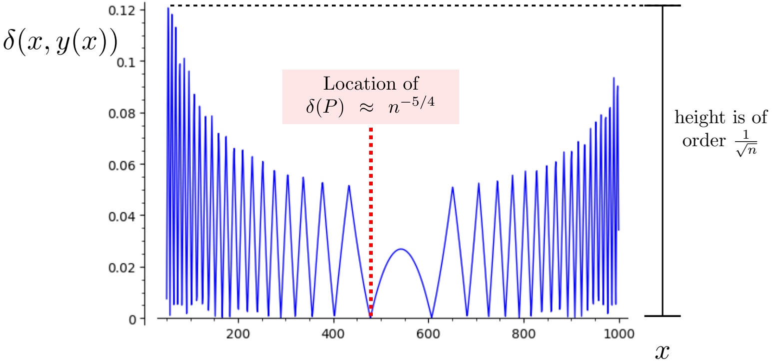
Location of the minimizer $y(x)$ for $n = 2000$



Semicircle shape is because of Brownian excursion.

Discrete pattern does not vanish in the limit.

Sorting probability $\delta(P)$ for $n = 2000$



Choosing x to be slightly left of midpoint gives smaller sorting probability because of zigzag pattern.

Back to second result

Theorem (C.-Pak-Panova '21)

For Catalan posets with n cells,

$$\delta(P_\lambda) \;\leq\; C n^{-\frac{5}{4}},$$

for some $C > 0$.

Important: Estimates are not done by NHLF, but by direct computation.

Better upper bound for general Young diagrams remain open.

What is next?

Theorem (C.-Pak-Panova '21)

Let $\lambda_1 \geq \ldots \geq \lambda_d \geq \varepsilon n$. For poset P_λ of Young diagram of λ, there exists x, y:

$$\delta(P_\lambda) \to 0 \qquad \text{as} \quad n \to \infty \, .$$

Open Problem

Prove same result for other families of posets, e.g., k-dimensional Young diagrams and periodic posets.

arXiv link: 2005.08390 and 2005.13686.

Webpage: http://math.ucla.edu/∼sweehong/

THANK YOU!

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