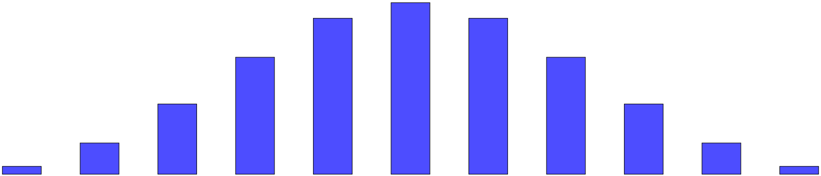
Log-concavity, Cross Product Conjectures, and FKG Inequalities in Order Theory

Swee Hong Chan

joint with Igor Pak and Greta Panova



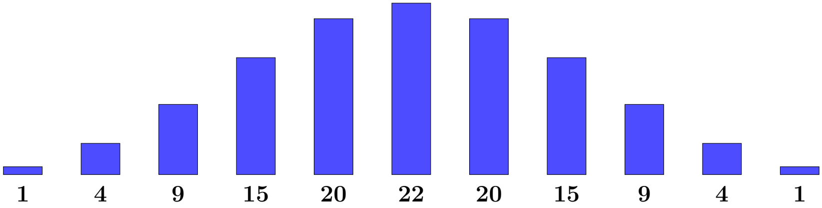
What is log-concavity?

A sequence $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ is log-concave if

$$a_k^2 \;\geq\; a_{k+1}\, a_{k-1} \qquad \text{for all} \;\; 1 < k < n.$$

Log-concavity (and positivity) implies unimodality:

$$a_1 \;\leq\; \cdots \;\leq\; a_m \;\geq\; \cdots \;\geq\; a_n \quad \text{for some} \;\; 1 \leq m \leq n.$$



Example: binomial coefficients

$$a_k \;=\; \binom{n}{k} \qquad k = 0, 1, \ldots, n.$$

This sequence is log-concave because

$$\frac{a_k^2}{a_{k+1}\,a_{k-1}} \;=\; \frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \;=\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right),$$

which is greater than 1.

Example: permutations with k inversions

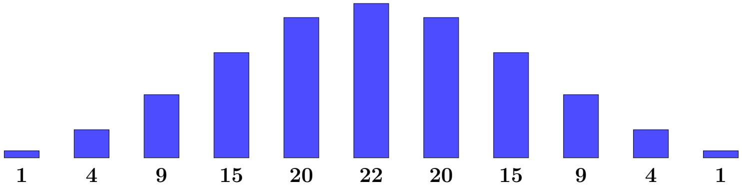
$$a_k \;=\; \text{number of } \pi \in S_n \text{ with } k \text{ inversions},$$

where inversion of π is pair $i < j$ s.t. $\pi_i > \pi_j$.

This sequence is log-concave because

$$\sum_{0 \le k \le \binom{n}{2}} a_k\, q^k \;=\; [n]_q! \;=\; \prod_{i=1}^{n-1} (1 + q + \ldots + q^i)$$

is a product of log-concave polynomials.

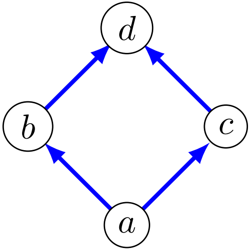


Log-concavity is a widespread phenomenon observed in numerous subjects in mathematics.

Today we focus on log-concavity for **probabilities in posets**.

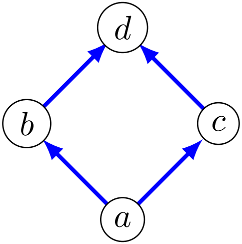
Partially ordered sets

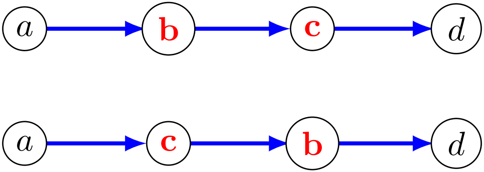
A poset P is a set X with partial order $\prec$ on X.



Linear extension

A linear extension L is a complete order of $\prec$.





We write $L(x) = k$ if x is k-th smallest in L.

Stanley's inequality

Fix $z \in P$.

$$N_k \text{ is probability that } \mathcal{L}(z) = k,$$

where $\mathcal{L}$ is uniform random linear extension of P.

Theorem (Stanley '81)

For every poset and $k \geq 1$,

$$N_k{}^2 \geq N_{k+1} N_{k-1}.$$

The inequality was initially conjectured by Chung-Fishburn-Graham, and was proved using Aleksandrov-Fenchel inequality for mixed volumes.

Our contribution

Problem

(Folklore, Graham '83, Biró-Trotter '11, Stanley '14)

Give a combinatorial proof of Stanley's inequality.

Answer (C.–Pak '21+)

More combinatorial proof for Stanley's inequality, with generalizations to weighted version.

Applications of log-concavity

$\frac{1}{3} - \frac{2}{3}$ Conjecture

Conjecture (Kislitsyn '68, Fredman '75, Linial '84)

For finite poset that is not completely ordered, there exist elements x, y:

$$\frac{1}{3} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{2}{3},$$

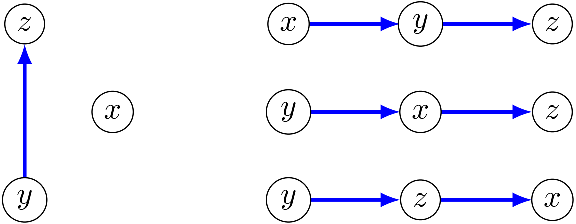
where $\mathcal{L}$ is uniform random linear extension of P.

Quote (Brightwell-Felsner-Trotter '95)

"This problem remains one of the most intriguing problems in the combinatorial theory of posets."

Why $\frac{1}{3}$ and $\frac{2}{3}$?

The upper,lower bound are achieved by this poset:



$$\mathbb{P}\big[\,\mathcal{L}(x) < \mathcal{L}(y)\,\big] \;=\; \frac{1}{3}\,; \qquad \mathbb{P}\big[\,\mathcal{L}(y) < \mathcal{L}(x)\,\big] \;=\; \frac{2}{3}\,.$$

The big breakthrough

Theorem (Kahn-Saks '84)

For poset that is not completely ordered, there exist elements x, y:

$$\frac{3}{11} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{8}{11},$$

roughly between 0.273 *and* 0.727.

Proof used log-concavity as a crucial component.

Proof sketch of Kahn-Saks Theorem

Find $x, y \in P$ such that

$$|h(y) - h(x)| \ \leq \ 1,$$

where $h(x) := \mathbb{E}[\mathcal{L}(x)]$ and $h(y) := \mathbb{E}[\mathcal{L}(y)]$.

Let F_k be probability that $\mathcal{L}(y) - \mathcal{L}(x) = k$.

$$\mathbb{P}\big[\,\mathcal{L}(x) < \mathcal{L}(y)\,\big] \ = \ F_1 + F_2 + \cdots + F_n,$$

$$\mathbb{P}\big[\,\mathcal{L}(y) < \mathcal{L}(x)\,\big] \ = \ F_{-1} + F_{-2} + \cdots + F_{-n}.$$

Proof sketch of Kahn-Saks Theorem

Since $|h(y) - h(x)|$ is small,

$$F_1 + 2F_2 + \cdots + nF_n \;\approx\; F_{-1} + 2F_{-2} + \cdots + nF_{-n}.$$

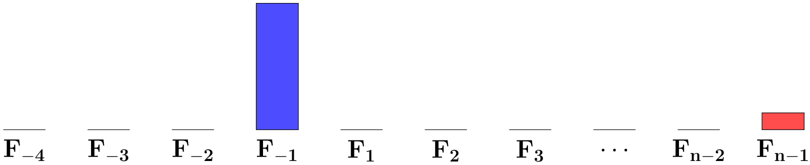
One can hope this implies

$$F_1 + F_2 + \cdots + F_n \;\approx\; F_{-1} + F_{-2} + \cdots + F_{-n},$$

which would then imply

$$\mathbb{P}[\,\mathcal{L}(x) < \mathcal{L}(y)\,] \;\approx\; \mathbb{P}[\,\mathcal{L}(y) < \mathcal{L}(x)\,] \;\approx\; 0.5.$$

But things can go really wrong:



Log-concavity comes to rescue

Theorem (Kahn–Saks '84)

For $k \neq 0$,

$$\begin{aligned} {F_k}^2 &\geq F_{k+1}\, F_{k-1}, \\ {F_{-k}}^2 &\geq F_{-(k+1)}\, F_{-(k-1)}. \end{aligned}$$

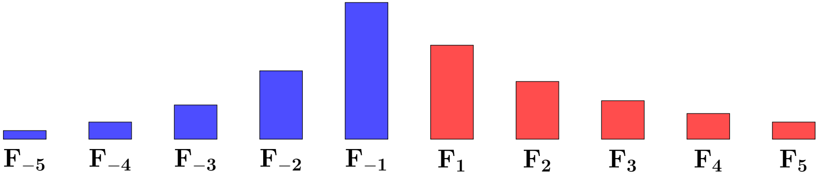
This generalizes Stanley's inequality, and was proved by Aleksandrov-Fenchel inequality.

Proof sketch of Kahn-Saks Theorem

Log-concavity (and other ineqs.) imply:

- $\mathbb{P}[\mathcal{L}(x) < \mathcal{L}(y)]$ is maximized (resp. minimized) when $F_1, F_2, \ldots, F_n$ is geometric sequence,

- $\mathbb{P}[\mathcal{L}(y) < \mathcal{L}(x)]$ is minimized (resp. maximized) when $F_{-1}, F_{-2}, \ldots, F_{-n}$ is geometric sequence.



Combined with $|h(y) - h(x)| \leq 1$, the result follows.

Best known bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture

Theorem (Brightwell-Felsner-Trotter '95)

For poset that is not completely ordered, there exist elements x, y:

$$\frac{5-\sqrt{5}}{10} \quad \leq \quad \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \quad \leq \quad \frac{5+\sqrt{5}}{10},$$

roughly between 0.276 *and* 0.724.

Note: Kahn–Saks bound was 0.273 and 0.727.

This bound cannot be improved for infinite posets.

Cross Product Conjecture

New ingredient: Cross Product Conjecture

Fix $x, y, z \in P$. Let $F(k, \ell)$ be probability that

$$\mathcal{L}(y) - \mathcal{L}(x) = k \quad \text{and} \quad \mathcal{L}(z) - \mathcal{L}(y) = \ell.$$

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k, \ell)\, F(k+1, \ell+1) \ \leq \ F(k+1, \ell)\, F(k, \ell+1).$$

Equivalently,

$$\det \begin{bmatrix} F(k, \ell) & F(k, \ell+1) \\ F(k+1, \ell) & F(k+1, \ell+1) \end{bmatrix} \ \leq \ 0.$$

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k,\ell)\,F(k+1,\ell+1) \;\leq\; F(k+1,\ell)\,F(k,\ell+1).$$

Brightwell-Felsner-Trotter proved the case $k = \ell = 1$ by Ahlswede–Daykin inequality.

Combined with Kahn–Saks proof, this gives the $\frac{5\pm\sqrt{5}}{10}$ bound for $\frac{1}{3} - \frac{2}{3}$ Conjecture.

What was known

Conjecture (Brightwell-Felsner-Trotter '95)

For $k, \ell \geq 1$,

$$F(k,\ell)\, F(k+1,\ell+1) \ \leq \ F(k+1,\ell)\, F(k,\ell+1).$$

Quote (Brightwell-Felsner-Trotter '95)

"Something more powerful seems to be needed to prove general form of Cross Product Conjecture."

Our results

Theorem 1 (C.-Pak-Panova '22)

Cross Product Conjecture is true for posets of width two.

Proved algebraically using matrix algebra argument and combinatorially through Lindström–Gessel–Viennot type argument.

Our results

Theorem 2 (C.-Pak-Panova '23+)
For every poset and $k, \ell \geq 1$,

$$F(k,\ell)\,F(k+1,\ell+1) \;<\; 2F(k+1,\ell)\,F(k,\ell+1).$$

Proof is based on Favard's inequality for mixed volumes, for which factor of 2 is tight for general geometric objects.

On the other hand, for specific classes of posets this factor of 2 can be improved.

A new protagonist

We now shift the attention

from linear extensions to **order-preserving maps**.

Order-preserving maps

Fix poset $P = (X, \prec)$ and integer $t \geq 1$.

A map $M : X \to \{1, \ldots, t\}$ is order-preserving if

$$x \prec y \qquad \text{implies} \qquad M(x) \leq M(y).$$

Linear extensions are order-preserving maps that are also bijections to $\{1, \ldots, |X|\}$.

Previously on linear extensions ...

- Log-concavity?
  Solved: Stanley '81, Kahn–Saks '84, C.-Pak
- Cross–product conjecture?
  Open: Brightwell–Felsner–Trotter '95, C.-Pak-Panova '22
- $\frac{1}{3}$–$\frac{2}{3}$ Conjecture?
  Open: Kahn–Saks '84, Brightwell–Felsner–Trotter '95

Can we **improve** on these results
for **order-preserving maps**?

Log-concavity for order-preserving maps

Graham's conjecture

Fix $z \in P$ and integer $t \geq 1$.

Let G_k be probability that $\mathcal{M}(z) = k$,

where $\mathcal{M}$ is uniform random ord.-pres. map $X \to [t]$.

Conjecture (Graham '83)

For every poset and $k \geq 1$,

$$G_k^2 \;\geq\; G_{k+1}\, G_{k-1}.$$

Graham's conjecture

Quote (Graham '83)

"It would seem that [*the conjecture*] *should have a proof based on the FKG or AD inequalities. However, such a proof has up to now* successfully eluded all attempts to find it*".*

What is Harris/FKG/AD inequalities?

They are fundamental inequalities in probability
that shows, in many random systems,
increasing events are positively correlated.

Example

For any $a, b, c, d \in V$ in Erdös–Renyi random graph,

$$\mathbb{P}\big[\, a \leftrightarrow b,\ c \leftrightarrow d \,\big] \ \geq \ \mathbb{P}\big[\, a \leftrightarrow b \,\big]\, \mathbb{P}\big[\, c \leftrightarrow d \,\big],$$

where $a \leftrightarrow b$ is event that a and b are connected.

Presence of one path increases probability of other path.

FKG or FGK?

Commun. math. Phys. 22, 89–103 (1971)

Correlation Inequalities on Some Partially Ordered Sets

C. M. Fortuin and P. W. Kasteleyn
Instituut-Lorentz, Rijksuniversiteit te Leiden, Leiden, Nederland

J. Ginibre
Laboratoire de Physique Théorique et Hautes Energies, Université de Paris-Sud
Orsay, France (Laboratoire associé au Centre National de la Recherche Scientifique)

Received March 2, 1971

Credit: Noga Alon, 2023

Graham's conjecture is true

Theorem (Daykin–Daykin–Paterson '84)

For every poset and $k \geq 1$,

$$G_k^2 \;\geq\; G_{k+1}\, G_{k-1}.$$

Proof used an explicit injective argument,
not based on FKG/AD inequality.

Quote (Daykin–Daykin–Paterson '84)

"[*Proof using FKG or Ahlswede–Daykin inequality*] *have as yet eluded discovery*".

Our results

Theorem 3 (C.–Pak ‘22+)

New proof of Daykin–Daykin–Paterson inequality based on Ahlswede–Daykin inequality, with generalization to multi-weighted version.

This proof validates Graham’s prediction.

Cross product conjecture for order-preserving maps

Our results

Fix $x, y, z \in P$ and $t \geq 1$. Let $G(k, \ell)$ be probability

$$\mathcal{M}(y) - \mathcal{M}(x) = k \;\text{ and }\; \mathcal{M}(z) - \mathcal{M}(y) = \ell,$$

where $\mathcal{M}$ is uniform random ord.-pres. map $X \to [t]$.

Theorem 4 (C.–Pak '22+)

For all integers k, ℓ,

$$G(k, \ell)\, G(k+1, \ell+1) \;\leq\; G(k+1, \ell)\, G(k, \ell+1).$$

This proves cross product conjecture for order-preserving maps.

Our results

Theorem (C.–Pak '22+)

For all integers k, ℓ,

$$G(k,\ell)\, G(k+1,\ell+1) \ \leq \ G(k+1,\ell)\, G(k,\ell+1).$$

Proof is based on same approach discovered when proving Daykin–Daykin–Paterson inequality.

This approach does not work for linear extensions, where inequality is known with factor of 2 in RHS.

Consequences: XYZ inequality

Theorem (Shepp '84)

For incomparable elements $x, y, z \in P$,

$$\mathbb{P}\big[\mathcal{L}(x) > \mathcal{L}(y) \mid \mathcal{L}(x) > \mathcal{L}(z)\big] \geq \mathbb{P}\big[\mathcal{L}(x) > \mathcal{L}(y)\big].$$

Inequalities remain true if random linear extension $\mathcal{L}$ is replaced with random order-preserving map $\mathcal{M}$.

XYZ inequality is a consequence of cross–product inequalities for order-preserving maps, with generalizations to weighted versions.

$\frac{1}{3}$–$\frac{2}{3}$ Conjecture for order-preserving maps

$\frac{1}{3}$–$\frac{2}{3}$ Conjecture for order-preserving maps

Conjecture

For finite poset that is not completely ordered, there exist elements x, y:

$$\frac{1}{3} \quad \leq \quad \lim_{t\to\infty} \mathbb{P}\big[\mathcal{M}_t(x) < \mathcal{M}_t(y)\big] \quad \leq \quad \frac{2}{3},$$

where $\mathcal{M}_t$ is uniform random o.p. map $X \to [t]$.

This is in fact equivalent to $\frac{1}{3}$–$\frac{2}{3}$ Conjecture for linear extensions.

All recent advances unfortunately do not improve known bounds for this conjecture.

Open problem

Kahn-Saks Conjecture

$\delta(P)$ is largest number such that there exist $x, y \in P$:

$$\delta(P) \ \leq \ \mathbb{P}\big[\mathcal{L}(x) < \mathcal{L}(y)\big] \ \leq \ 1 - \delta(P).$$

Note that $\frac{1}{3} - \frac{2}{3}$ Conjecture is equivalent to $\delta(P) \geq \frac{1}{3}$ for P not completely ordered.

Conjecture (Kahn-Saks '84)

$$\delta(P) \to \frac{1}{2} \quad \textit{as} \quad \text{width}(P) \to \infty.$$

Kahn-Saks Conjecture

Conjecture (Kahn-Saks '84)

$$\delta(P) \to \frac{1}{2} \quad \textit{as} \quad \mathrm{width}(P) \to \infty.$$

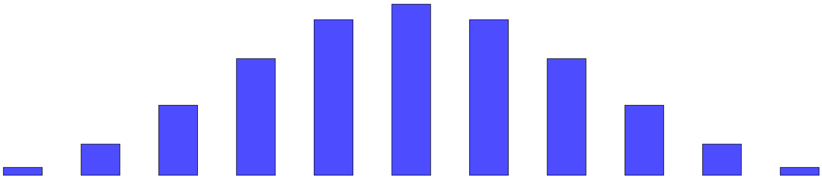
Komlós '90 proved Conjecture for posets with $\Omega(\frac{n}{\log\log\log n})$ minimal elements.

C.-Pak-Panova '21 proved Conjecture for Young diagram posets with fixed width.

THANK YOU!

Webpage: www.math.rutgers.edu/~sc2518/

Email: sc2518@rutgers.edu



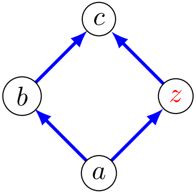
Order-reversing weight

A weight $\omega : X \to \mathbb{R}_{>0}$ is order-reversing if

$$\omega(x) \ \geq \ \omega(y) \qquad \text{whenever} \qquad x \prec y.$$

Weight of linear extension L is

$$\omega(L) \ := \prod_{L(x)<L(z)} \omega(x).$$



$$\omega(L) = \omega(a)$$

$$\omega(L) = \omega(a)\omega(b)$$

Weighted Stanley's inequality

Let $N_{\omega,k}$ be probability that $\mathcal{L}(z) = k$,

where $\mathcal{L}_\omega$ is ω-weighted random linear extension.

Theorem 5 (C.–Pak '21+)

For every poset and $k \geq 1$,

$$N_{\omega,k}{}^2 \;\geq\; N_{\omega,k+1}\, N_{\omega,k-1}.$$

Proof used combinatorial atlas method,

a new tool to establish log-concave inequalities.