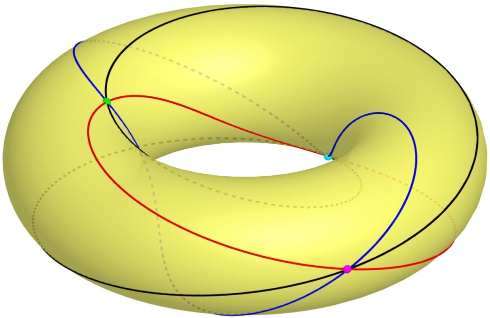
Toric arrangements that come from graphs

Swee Hong Chan
Joint work with Marcelo Aguiar
Cornell University

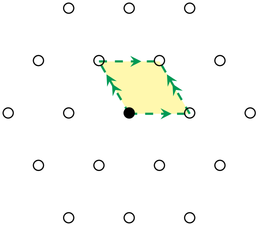


Torus

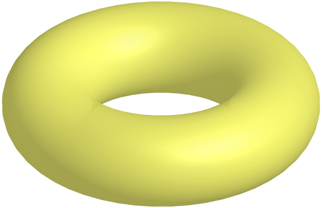
Let V be a real vector space.

A lattice L is the additive subgroup of V generated by a basis of V.

The associated torus is the quotient group $T := V/L$.



L



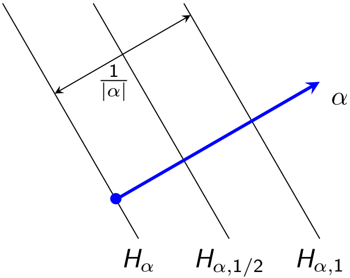
T

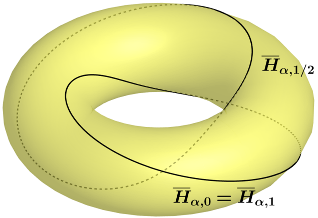
Hyperplanes and hypertori

Fix an inner product $\langle\ ,\ \rangle$ on V.

For each nonzero vector $\alpha \in V$ and $k \in \mathbb{R}$, let

$$\begin{aligned} H_\alpha &:= \{x \in V \mid \langle \alpha, x\rangle = 0\} \quad \text{(linear hyperplane)}; \\ H_{\alpha,k} &:= \{x \in V \mid \langle \alpha, x\rangle = k\} \quad \text{(affine hyperplane)}; \\ \overline{H}_{\alpha,k} &:= \text{the image of } H_{\alpha,k} \text{ in the torus } T \quad \text{(hypertorus)}. \end{aligned}$$

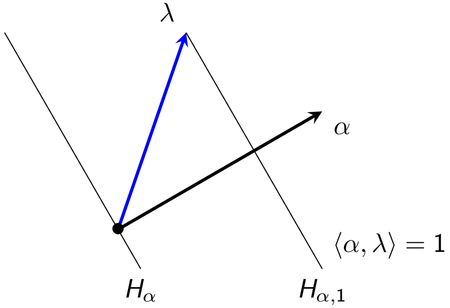


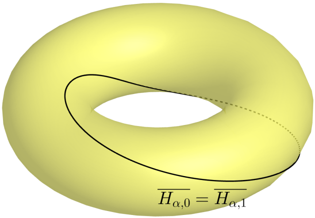


Integrality

A vector $\alpha \in V$ is *L*-integral if $\langle \alpha, \lambda \rangle \in \mathbb{Z}$ for all $\lambda \in L$.

If α is *L*-integral, then $\{\overline{H}_{\alpha,k} \mid k \in \mathbb{Z}\}$ is a finite set.

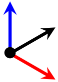




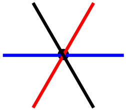
Arrangements

Let X be a finite set of nonzero L-integral vectors in V.

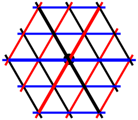
$$\mathcal{A}(X) := \{H_\alpha \mid \alpha \in X\} \text{ (linear arrangement)};$$
$$\widetilde{\mathcal{A}}(X) := \{H_{\alpha,k} \mid \alpha \in X,\ k \in \mathbb{Z}\} \text{ (affine arrangement)};$$
$$\overline{\mathcal{A}}(X, L) := \{\overline{H}_{\alpha,k} \mid \alpha \in X,\ k \in \mathbb{Z}\} \text{ (toric arrangement)}.$$



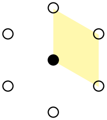
X



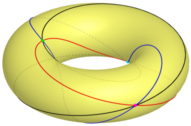
$\mathcal{A}(X)$



$\widetilde{\mathcal{A}}(X)$



L



$\overline{\mathcal{A}}(X, L)$

Root systems

Let X be a subset of a crystallographic root system Φ in V.

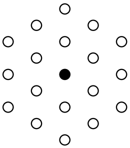
Two lattices associated to Φ:

$$\mathbb{Z}\Phi^\vee := \mathbb{Z}\{2\beta/\langle\beta,\beta\rangle \mid \beta \in \Phi\} \quad \text{(coroot lattice)};$$

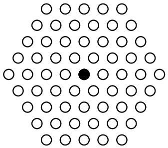
$$\widehat{\mathbb{Z}\Phi} := \{\alpha \in V \mid \langle\alpha,\lambda\rangle \in \mathbb{Z} \text{ for all } \lambda \in \Phi\} \quad \text{(coweight lattice)}.$$

Then X is integral with respect to either lattice.

A_2



$\mathbb{Z}A_2^\vee$



$\widehat{\mathbb{Z}A_2}$

One graph, two toric arrangements

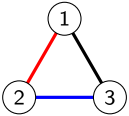
Let G be a simple connected graph with vertex set $[n]$.
Let $A_{n-1} := \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$ (root system of type A).

View G as this finite subset of A_{n-1}:

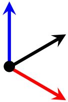
$$\{e_i - e_j \mid \{i,j\} \text{ is an edge of } G \text{ and } i < j\}.$$

There are two kinds of toric graphic arrangements:

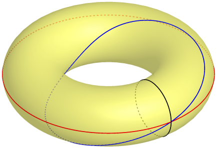
- $\overline{\mathcal{A}}(G, \widehat{\mathbb{Z}A_{n-1}})$, the coweight graphic arrangement.
- $\overline{\mathcal{A}}(G, \mathbb{Z}A_{n-1}^{\vee})$, the **coroot graphic arrangement**.



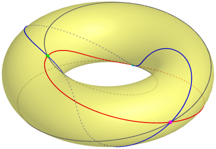
K_3



"K_3"

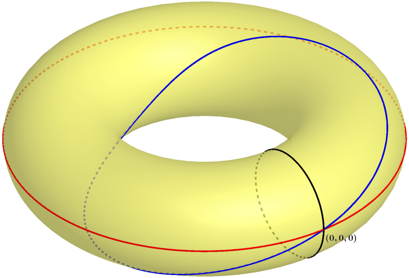


$\overline{\mathcal{A}}(G, \widehat{\mathbb{Z}A_{n-1}})$

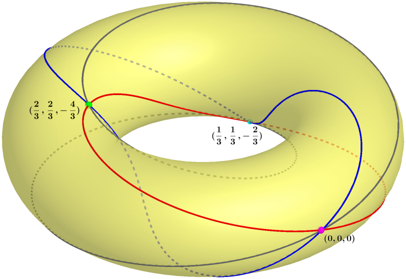


$\overline{\mathcal{A}}(G, \mathbb{Z}A_{n-1}^{\vee})$

The coweight and coroot arrangement for K_3



Coweight arrangement



Coroot arrangement

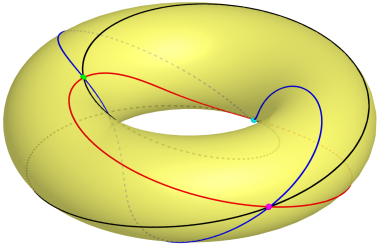
$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\};$$

$$L = \begin{cases} \langle (1,-1,0), (0,1,-1), (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}) \rangle_{\mathbb{Z}} & \text{(coweight lattice);} \\ \langle (1,-1,0), (0,1,-1) \rangle_{\mathbb{Z}} & \text{(coroot lattice).} \end{cases}$$

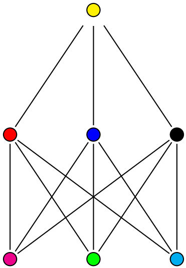
Flats

A flat of a toric arrangement $\overline{\mathcal{A}}$ is a connected component of an intersection of hypertori in $\overline{\mathcal{A}}$.

The intersection poset $\overline{\Pi}(\overline{\mathcal{A}})$ is the (po)set of flats of $\overline{\mathcal{A}}$, ordered by inclusion.



$\overline{\mathcal{A}}$



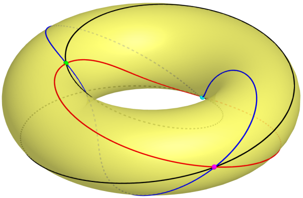
$\overline{\Pi}(\overline{\mathcal{A}})$

Flats of coroot graphic arrangements

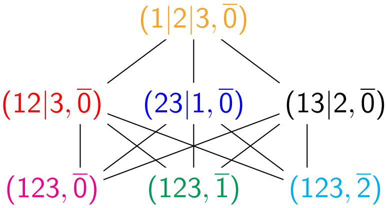
Theorem

- *Each flat of the coroot arrangement corresponds to* $(B, \bar{j})$,
    - B *is a bond of the graph* G;
    - $\bar{j} \in \mathbb{Z}_{g(B)}$, *with* $g(B)$ *is the gcd of the size of blocks of* B.
- *Inclusion of flats corresponds to the partial order:*

$$(B, \bar{j}) \leq (B', \overline{j'}) \iff B' \text{ refines } B \text{ and } \bar{j} \equiv \overline{j'} \mod g(B').$$



Coroot argmnt of K_3



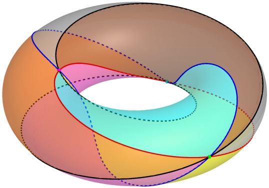
Coroot intersection poset of K_3

Characteristic polynomial

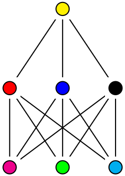
The characteristic polynomial of a toric arrangement $\overline{\mathcal{A}}$ is

$$\overline{\chi}(\overline{\mathcal{A}};\, t) := \sum_{F \in \overline{\Pi}(\overline{\mathcal{A}})} \underbrace{\mu(F, T)}_{\text{Möbius function}} \quad t^{\dim F}.$$

The number of regions in $\overline{\mathcal{A}}$ is equal to $|\overline{\chi}(\overline{\mathcal{A}};\, 0)|$.
(Novik Postnikov Sturmfels '02, Reiner, Ehrenborg Readdy Sloane '09)



$\overline{\mathcal{A}}$



$\overline{\Pi}$

$$2+2+2-t-t-t+t^2$$
$$= t^2 - 3t + 6$$

$\overline{\chi}(\overline{\mathcal{A}};\, t)$

Examples of coroot characteristic polynomials

- For the path graph P_n,

$$\overline{\chi}(P_n, \mathbb{Z}A_{n-1}^{\vee};\, t) = (-1)^{n-1} \sum_{d|n} \varphi(d)(1-t)^{\frac{n}{d}-1},$$

  where φ is Euler's totient function.

- For the star graph $K_{1,n-1}$,

$$\overline{\chi}(K_{1,n-1}, \mathbb{Z}A_{n-1}^{\vee};\, t) = (t-1)^{n-1} + (-1)^{n-1}(n-1).$$

- For the complete graph K_n,

$$\overline{\chi}(K_n, \mathbb{Z}A_{n-1}^{\vee};\, t) = (-1)^{n-1}(n-1)! \sum_{d|n} (-1)^{\frac{n}{d}-1} \varphi(d) \binom{\frac{t}{d}-1}{\frac{n}{d}-1}$$

(Ardila Castillo Henley '15)

Divisible colorings

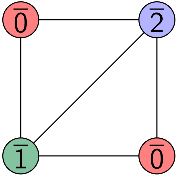
A proper divisible m-coloring of G is a function $f : V \to \mathbb{Z}_m$ with

- $f(i) \neq f(j)$ if i and j are adjacent in G; and
- $\sum_{i \in V} f(i) \equiv 0 \pmod{m}$.

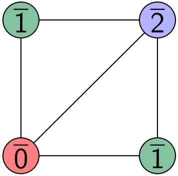
Theorem

For any positive multiple m of n,

$$\#\text{ of proper divisible } m\text{-colorings of } G = |\overline{\chi}(G, \mathbb{Z}A_{n-1}^{\vee};\ m)|.$$



Divisible 3-coloring

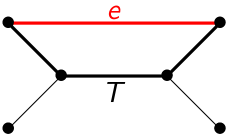


Non-divisible 3-coloring

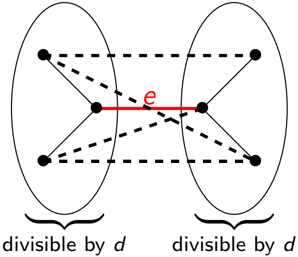
Divisible activities

Fix a total order on $E(G)$. For any $d \in \mathbb{N}$ and any sp. tree T,

- An edge e is externally active w.r.t T if $e \notin T$ and e is the minimum edge in $\mathrm{Cycle}(T, e)$.



- An edge e is d-internally active w.r.t T if $e \in T$ and e is the minimum edge in $\text{d-Cut}(T, e)$.



A formula for the coroot characteristic polynomial

Theorem

The coroot characteristic polynomial of G is equal to

$$\overline{\chi}(G, \mathbb{Z}A_{n-1}^{\vee};\, t) = (-1)^{n-1} \sum_{d|n} \varphi(d)\, \overline{\chi}_d(G; t),$$

where φ is the Euler's totient function, and

$$\overline{\chi}_d(G; t) := \sum_{\substack{T \text{ spanning tree of } G \\ \text{with ext. activity } 0}} (1-t)^{int_d(T)}.$$

Other properties of the coroot polynomial

- The evaluation at $t = 0$ is the number of acyclic orientations of G with exactly one sink.
- The polynomial satisfies a variant of deletion-contraction recurrence.
- There are extensions of these properties to the arithmetic Tutte polynomial from (Moci '12).

Voronoi cells and confinement

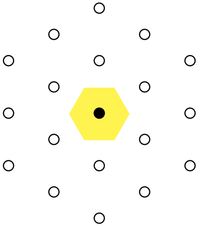
The Voronoi cell of a lattice L is

$$\mathrm{Vor}(L) := \{x \in V \mid \ |x| < |x - \lambda| \text{ for all } \lambda \in L\}.$$

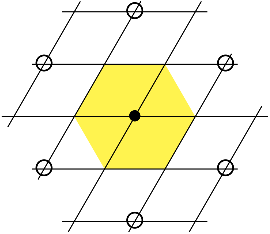
$\mathrm{Vor}(L)$ is a fundamental domain for the torus V/L.

A set of vectors X confines L if

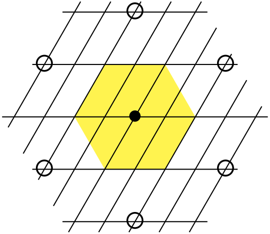
$$\forall\, H \in \widetilde{\mathcal{A}}(X), \qquad H \cap \mathrm{Vor}(L) \neq \emptyset \Rightarrow 0 \in H.$$



$\mathrm{Vor}(L)$



Confined



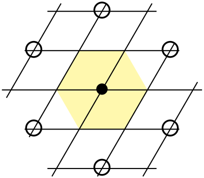
Not confined

Voronoi equivalence

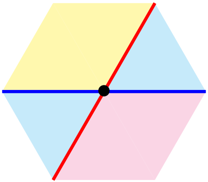
Assume X is L-integral, spanning, and confines L.

Two regions of $\mathcal{A}(X)$ are Voronoi equivalent if their restriction to Vor(L) are projected to the same toric region of $\overline{\mathcal{A}}(X, L)$.

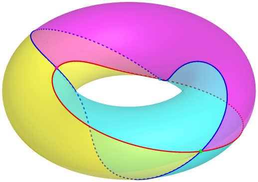
The equivalence classes correspond to regions of $\overline{\mathcal{A}}(X, L)$.



Vor(L) and $\widetilde{\mathcal{A}}(X)$



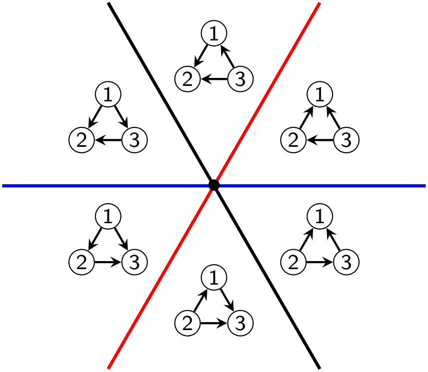
$\mathcal{A}(X)$



$\overline{\mathcal{A}}(X, L)$

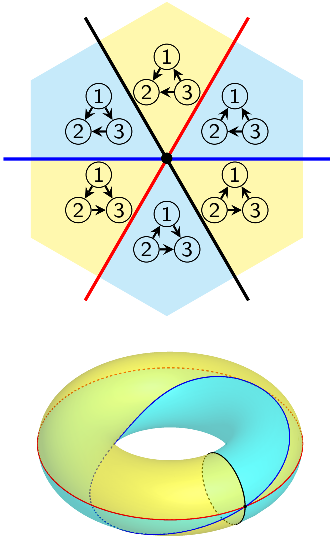
Acyclic orientations

Each region of of the linear graphic arrangement corresponds to an acyclic orientation of the graph.
(Greene Zaslavsky '83)

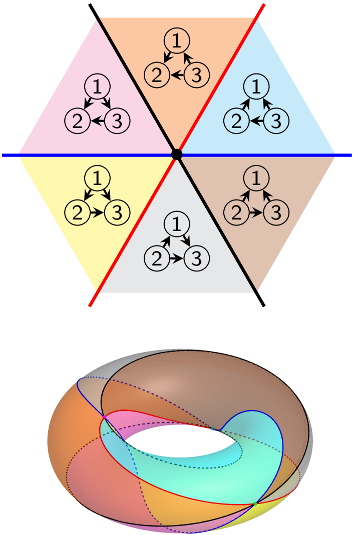


$\mathcal{A}(K_3)$

Graphic Voronoi equivalence



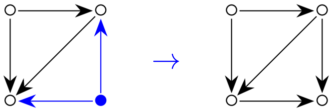
Coweight Voronoi equivalence



Coroot Voronoi equivalence

Voronoi equivalence for coweight toric arrangements

A source-to-sink flip (Mosesjan '72, Pretzel '86) is



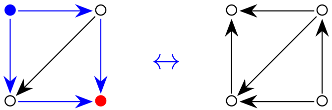
Theorem

Two acyclic orientations are coweight Voronoi equivalent iff one can reach the other by successive source-to-sink flips.

A similar observation for toric partial orders was made by Develin Macauley Reiner '16.

Voronoi equivalence for coroot toric arrangements

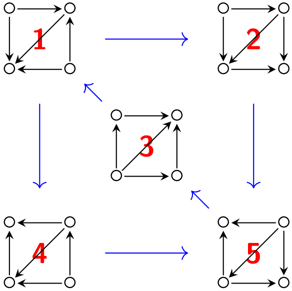
A source-sink exchange is



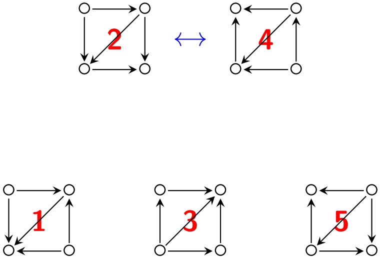
Theorem

Two acyclic orientations are coroot Voronoi equivalent iff one can reach the other by successive source-sink exchanges.

Comparison between two relations



One equivalence class under the coweight relation



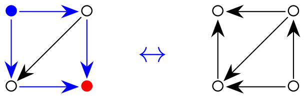
Four equivalence classes under the coroot relation

Voronoi equivalence for coroot toric arrangements

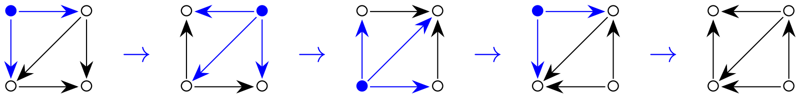
Theorem

For any two acyclic orientations, TFAE:

- *They are coroot Voronoi equivalent.*
- *One can reach another by successive source-to-sink exchanges.*
- *One can reach another by successive n-step source-to-sink flips.*



A source-sink exchange.



An n-step source-to-sink flip.

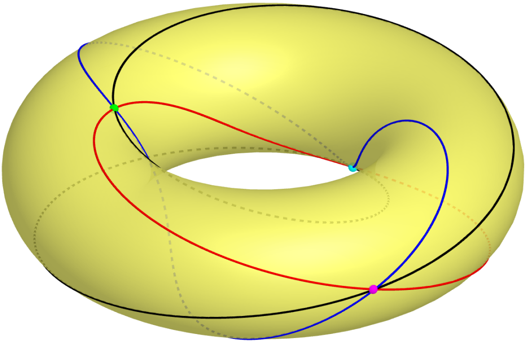
Number of orientations in one class

- The path P_n has n coroot Voronoi equivalence classes,

$$\begin{array}{l}\#\text{ of orientations}\\ \text{in the } i\text{-th class}\end{array} = \frac{1}{2}\left|\left\{ S \subseteq [n] \;\middle|\; \sum_{x \in S} x \equiv i \mod n \right\}\right|.$$

- The cycle C_n has $(n-1)n$ coroot Voronoi equivalence classes,

$$\begin{array}{l}\#\text{ of orientations}\\ \text{in the } (i,j)\text{-th class}\end{array} = \left|\left\{ S \subseteq [n] \;\middle|\; |S| = i, \sum_{x \in S} x \equiv j \mod n \right\}\right|.$$



THANK YOU!

Extended abstract can be found at:

`https://www.math.cornell.edu/~sc2637/`