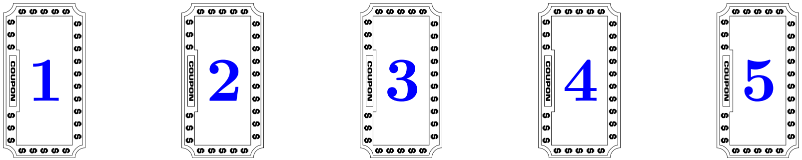
Coupon collector's problem



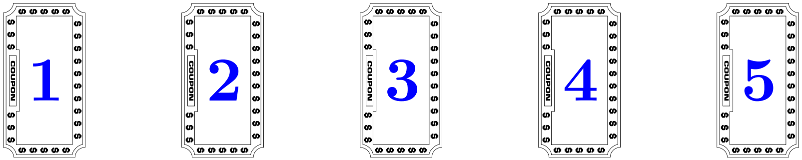
Coupon collector's problem

There are n different types of coupons.

Each purchase comes with one random coupon.

Every coupon is equally likely to appear.

How many purchases needed to collect all coupons?



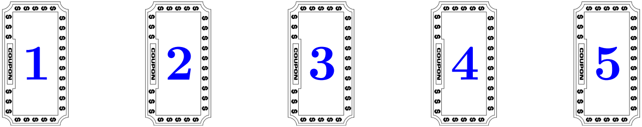
No definite answer

The answer is random:

- If lucky, then n purchases are enough.
- If unlucky, then no purchases will be enough.

More suitable question:

How many purchases needed on average to collect all coupons?



Expected value

X is a discrete random variable with real number outcomes $x_1, \ldots, x_k, \ldots$.

Expected value of X is

$$E[X] \ = \ x_1 P(X = x_1) + \ldots + x_k P(X = x_k) + \ldots,$$

the value of X on average if same experiment is repeated over and over again.

Note: Assume this infinite sum is well-defined.

Expected value: example

Play the following coin flipping game:

- Win 1 dollar if coin comes out head,
- Lose 1 dollar if coin comes out tail.

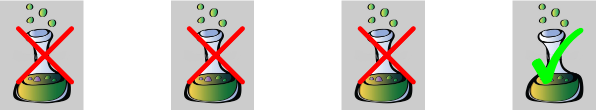
X is the amount of money won from this game.

$$
\begin{aligned}
E[X] &= 1 \times P(X = 1) + (-1) \times P(X = -1) \\
&= 1 \times 0.5 \, + \, (-1) \times 0.5 \; = \; 0.
\end{aligned}
$$

Geometric random variable

- You perform a sequence of independent trials.
- Each trial has two outcomes: success or failure.
- Probability of success for every trial is p.

Geometric random variable X is total number of trials needed until first success.



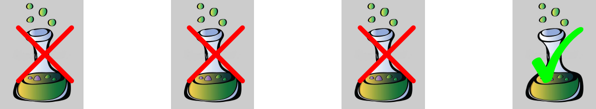
Geometric random variable: probabilities

For the first success to be at k-th trial:

- First $(k-1)$ trials are failures,
- The k-th trial is success.

Probability for this event is

$$
\begin{aligned}
P(X = k) &= \underbrace{(1-p) \times \ldots \times (1-p)}_{(k-1)\ \text{times}} \times p \\
&= (1-p)^{k-1}\, p.
\end{aligned}
$$



Geometric random variable: expected value

Expected value $E[X]$ is equal to

$$P(X=1) + 2\,P(X=2) + \ldots + k\,P(X=k) + \ldots$$

Plugging in the probabilities:

$$\begin{aligned} E[X] &= p + \ldots + k\,(1-p)^{k-1}\,p + \ldots \\ &= p\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p}. \end{aligned}$$

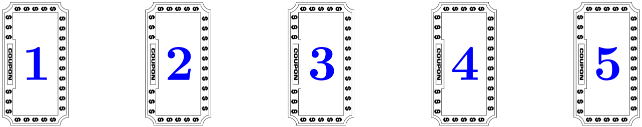
Note: This is derivative of geometric series.

Note: Write $E[X] = \frac{1}{p}$ on separate piece of paper.

Back to coupon collector

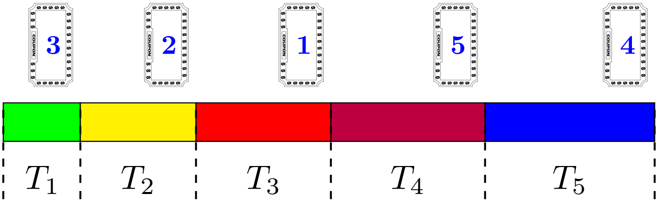
T = number of purchases needed to collect all coupons.

Coupon collector's problem: Compute $E[T]$.



Change of perspective

- $T_1 \;=\;$ purchases needed to get 1 coupon.
- $T_2 \;=\;$ purchases needed to get 2 coupons after collecting 1 coupon.
- $T_k \;=\;$ purchases needed to get k coupons after collecting $k-1$ coupons.



Linearity of expectation

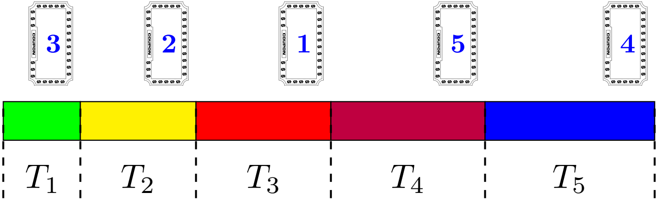
Then to collect all n coupons, we need

$$T = T_1 + T_2 + \ldots + T_n.$$

So

$$E[T] = E[T_1] + E[T_2] + \ldots + E[T_n].$$

This is called linearity of expectations.



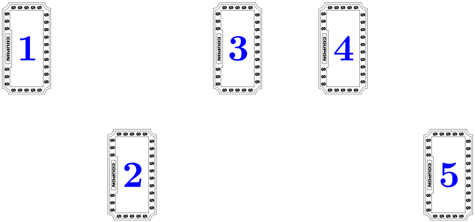
Geometric random variable comes back

After collecting $k - 1$ coupons, for next purchase:

- Failure = getting coupon you already have.
  There are $k - 1$ choices.
- Success = getting a new coupon.
  There are $n - k + 1$ choices.

Have:

Don't Have:



Geometric random variable comes back

Then T_k is geometric random variable with

$$\text{success probability} \quad p = \frac{n-k+1}{n}.$$

So $$E[T_k] \;=\; \frac{1}{p} \;=\; \frac{n}{n-k+1}.$$

Solution to coupon's collector problem

$$
\begin{aligned}
E[T] &= E[T_1] + E[T_2] + E[T_3] + \ldots + E[T_n] \\
&= \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{1} \\
&= n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{1}\right) \\
&= n\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right).
\end{aligned}
$$

This is harmonic series, and

$$
E[T] \quad \text{is approximately} \quad n\int_1^n \frac{1}{x}\,dx \quad = \quad n\log n.
$$

Values of $E[T]$

- Number of coupons $n = 5$;
  Average purchases needed $E[T] = 11.4$.
- Number of coupons $n = 25$;
  Average purchases needed $E[T] = 95.4$.
- Number of coupons $n = 100$;
  Average purchases needed $E[T] = 518.7$.

How close is your guess to the answer?

THANK YOU!