Math 170S

Lecture Notes Section 8.8 *†

Likelihood ratio tests

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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†This notes is based on Hanbaek Lyu's and Liza Rebrova's notes from the previous quarter, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Example: MLE, Level 0

Let X be a normal random variable with unknown mean μ and **known variance** 1. Suppose that the samples for X are

$$x_1 = 1.1; \qquad x_2 = -1.1; \qquad x_3 = 2; \qquad x_4 = -3.$$

Suppose that you are being asked to guess if

- $\mu = 1$; or
- $\mu = 3$.

Which one would be the most rational choice?

2 Recap: likelihood function

Let X be an RV with density f_θ, with **unknown parameter** θ. The **likelihood function** is

$$L(\theta) := f_\theta(x_1)\, f_\theta(x_2) \ldots f_\theta(x_n).$$

The larger the likelihood function, the more likely that θ is the correct choice.

3 Answer: MLE, Level 0

The density function for the normal random variable is

$$f_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

So the likelihood function is (write this down)

$$\begin{aligned}
L(\mu) &= f_\mu(x_1) \ldots f_\mu(x_n) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_1-\mu)^2}{2}\right) \ldots \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_n-\mu)^2}{2\sigma^2}\right) \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left[(x_1-\mu)^2 + \ldots + (x_n-\mu)^2\right]\right).
\end{aligned}$$

Since $\sigma^2 = 1$, we have

$$L(\mu) = \left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{1}{2}\left[(x_1-\mu)^2 + \ldots + (x_n-\mu)^2\right]\right).$$

So we have

$$\begin{aligned} L(1) &= \left(\frac{1}{2\pi}\right)^{4/2} \exp\left(-\frac{1}{2}\left[(1.1-1)^2 + \ldots + (-3-1)^2\right]\right) \\ &= 5.7 \times 10^{-7}; \\ L(3) &= \left(\frac{1}{2\pi}\right)^{4/2} \exp\left(-\frac{1}{2}\left[(1.1-3)^2 + \ldots + (-3-3)^2\right]\right) \\ &= 8.6 \times 10^{-15}. \end{aligned}$$

Since $L(1) > L(3)$, the rational choice here is $\mu = 1$.

4 Example: MLE, Level 1

Let X be a normal RV with unknown mean μ and **known variance** 1, with samples

$$x_1 = 1.1; \qquad x_2 = -1.1; \qquad x_3 = 2; \qquad x_4 = -3.$$

Suppose that you are being asked to guess if

- $\mu = 1$; or
- $\mu \neq 1$. (**Notice the difference here**.)

Which one would be the most rational choice?

5 Answer: MLE, Level 1

For the case $\mu = 1$, the likelihood function is

$$
\begin{aligned}
L(1) &= \left(\frac{1}{2\pi}\right)^{4/2} \exp\left(-\frac{1}{2}\left[(1.1-1)^2 + \ldots + (-3-1)^2\right]\right) \\
&= 5.7 \times 10^{-7}.
\end{aligned}
$$

For the second case $\mu \neq 1$, which values of μ should we choose to substitute into $L(\mu)$?

Answer: Use MLE $\widehat{\mu}$.

- Recall that the MLE $\widehat{\mu}$ is the value that maximizes the likelihood function L;
- Also recall that, for normal random variables, $\widehat{\mu}$ is equal to the sample mean $\overline{\mathrm{x}}$.

Our sample mean here is equal to $\overline{\mathrm{x}} = -\frac{1}{4}$, so the likelihood function for $\mu \neq 1$ is

$$\begin{aligned} L(-\frac{1}{4}) &= \left(\frac{1}{2\pi}\right)^{4/2} \exp\Bigg(-\frac{1}{2}\Big[(1.1-(-\frac{1}{4}))^2 + \\ &\quad (-1.1-(-\frac{1}{4}))^2 + (2-(-\frac{1}{4}))^2 + (-3-(-\frac{1}{4}))^2\Big]\Bigg) \\ &= 1 \times 10^{-5}. \end{aligned}$$

Since $L(-\frac{1}{4}) \ > \ L(1)$, the rational choice here is $\mu \neq 1$.

6 Settings: likelihood ratio tests

Object: X is a random variable with density function f_θ with **unknown** θ.

Hypotheses:

- H_0: θ comes from a given set ω.
- H_1: θ does not come from ω.

Input: Random samples $X_1, \ldots, X_n$ for X and significance level α.

Methodology:

- Find the value $\widehat{\theta}_0$ that maximizes the likelihood function L **among the θ's in given set ω**.

- Find the value $\widehat{\theta}_1$ that maximizes the likelihood function L **among all θ**. (This means $\widehat{\theta}_1$ is the maximum likelihood estimate.)

- Reject H_0 if
$$\frac{L(\widehat{\theta}_0)}{L(\widehat{\theta}_1)} \leq k.$$
Do not reject H_0 otherwise.

- Here $k \in [0, 1]$ is a number chosen so that the test has significance level α.

7 Example: likelihood ratio test, Level 1

Let X be a normal RV with unknown mean μ and **known variance** 1, with samples

$$x_1 = 1.1; \qquad x_2 = -1.1; \qquad x_3 = 2; \qquad x_4 = -3.$$

The hypothesis are:

- The null hypothesis H_0 is $\mu \in \{1, 2\}$.
- The alternative hypothesis H_1 is $\mu \notin \{1, 2\}$.

Suppose that we reject H_0 if $\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} \leq 0.3$. Should we reject H_0 or not?

8 Answer: likelihood ratio test, Level 1

We first compute the ratio $\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)}$:

$$\begin{aligned}
\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} &= \frac{\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \widehat{\mu}_0)^2\right]\right)}{\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \widehat{\mu}_1)^2\right]\right)} \\
&= \frac{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \widehat{\mu}_0)^2\right]\right)}{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \widehat{\mu}_1)^2\right]\right)}
\end{aligned}$$

Now note that (BT) (write this down)

$$
\begin{aligned}
\sum_{i=1}^{n}(x_i - \widehat{\mu}_0)^2 \ &= \ \sum_{i=1}^{n}((x_i - \overline{\mathrm{x}}) + (\overline{\mathrm{x}} - \widehat{\mu}_0))^2 \\
&= \ \sum_{i=1}^{n}\left[(x_i - \overline{\mathrm{x}})^2 + 2(x_i - \overline{\mathrm{x}})(\overline{\mathrm{x}} - \widehat{\mu}_0) + (\overline{\mathrm{x}} - \widehat{\mu}_0)^2\right] \\
&= \ \sum_{i=1}^{n}(x_i - \overline{\mathrm{x}})^2 \ + \ 2(\overline{\mathrm{x}} - \widehat{\mu}_0)\sum_{i=1}^{n}(x_i - \overline{\mathrm{x}}) \ + \ \sum_{i=1}^{n}(\overline{\mathrm{x}} - \widehat{\mu}_0)^2 \\
&= \ \sum_{i=1}^{n}(x_i - \overline{\mathrm{x}})^2 \quad + \quad 2(\overline{\mathrm{x}} - \widehat{\mu}_0)(n\,\overline{\mathrm{x}} - n\,\overline{\mathrm{x}}) \quad + \quad n(\overline{\mathrm{x}} - \widehat{\mu}_0)^2 \\
&= \ \sum_{i=1}^{n}(x_i - \overline{\mathrm{x}})^2 \quad + \quad n(\overline{\mathrm{x}} - \widehat{\mu}_0)^2.
\end{aligned}
$$

So we have (write this down)

$$\begin{aligned}\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} &= \frac{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^n (x_i - \widehat{\mu}_0)^2\right]\right)}{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^n (x_i - \widehat{\mu}_1)^2\right]\right)} \\ &= \frac{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^n (x_i - \overline{\mathrm{x}})^2 \ + \ n(\overline{\mathrm{x}} - \widehat{\mu}_0)^2\right]\right)}{\exp\left(-\frac{1}{2}\left[\sum_{i=1}^n (x_i - \widehat{\mu}_1)^2\right]\right)}\end{aligned}$$

Now recall that $\widehat{\mu}_1 = \overline{\mathrm{x}}$ (sample mean is the the MLE for normal RV).

Also note that $n = 4$ and $\sigma^2 = 1$. This means that

$$\begin{aligned}\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} &= \frac{\exp\left(-\frac{1}{2}\left[\sum_{i=1}^4 (x_i - \overline{\mathrm{x}})^2 \ + \ 4(\overline{\mathrm{x}} - \widehat{\mu}_0)^2\right]\right)}{\exp\left(-\frac{1}{2}\left[\sum_{i=1}^4 (x_i - \overline{\mathrm{x}})^2\right]\right)} \\ &= \exp\left(-2(\overline{\mathrm{x}} - \widehat{\mu}_0)^2\right).\end{aligned}$$

We now compute $\widehat{\mu}_0$.

Note that the set ω for the null hypothesis is $\omega = \{1, 2\}$.

We check the value of $L(\mu)$ for these two values

$$\begin{aligned} L(1) &= \left(\frac{1}{2\pi}\right)^{4/2} \exp\left(-\frac{1}{2}\left[(1.1-1)^2 + \ldots + (-3-1)^2\right]\right) \\ &= 5.7 \times 10^{-7}; \\ L(2) &= \left(\frac{1}{2\pi}\right)^{4/2} \exp\left(-\frac{1}{2}\left[(1.1-2)^2 + \ldots + (-3-2)^2\right]\right) \\ &= 5.2 \times 10^{-10}. \end{aligned}$$

So the winner is $\widehat{\mu}_0 = 1$. Hence we have

$$\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} = \exp\left(-2(\overline{\mathrm{x}} - \widehat{\mu}_0)^2\right) = \exp\left(-2\left(\frac{-1}{4} - 1\right)^2\right)$$
$$= 0.04.$$

This ratio is smaller than 0.3, so we reject H_0.

9 Example: likelihood ratio test, Level 2

Let X be a normal random variable with unknown mean μ and **known variance** σ^2 (**notice the difference here**).

- H_0 is $\mu = \mu_0$.
- H_1 is $\mu \neq \mu_0$. (**Notice the difference here**.)

Suppose that we reject H_0 if $\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} \leq k$.

Determine the value of k so that the significance level of this test is equal to α.

10 Answer: likelihood ratio test, Level 2

The ratio $\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)}$ is equal to (see what you previously wrote)

$$\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} = \frac{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(X_i-\overline{\mathrm{X}})^2 \ + \ n(\overline{\mathrm{X}}-\widehat{\mu}_0)^2\right]\right)}{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(X_i-\widehat{\mu}_1)^2\right]\right)}$$

Now note that $\widehat{\mu}_0 = \mu_0$ since $\omega = \{\mu_0\}$. Also note that $\widehat{\mu}_1 = \overline{\mathrm{X}}$ since the MLE for normal random variable is sample mean.

So we have

$$\begin{aligned}\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} &= \frac{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(X_i-\overline{\mathrm{X}})^2 \ + \ n(\overline{\mathrm{X}}-\mu_0)^2\right]\right)}{\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(X_i-\overline{\mathrm{X}})^2\right]\right)} \\ &= \exp\left(-\frac{n}{2\sigma^2}\left(\overline{\mathrm{X}}-\mu_0\right)^2\right).\end{aligned}$$

We then have (BT)

$$
\begin{aligned}
\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} &\leq k \\
\exp\left(-\frac{n}{2\sigma^2}\left(\overline{\mathrm{X}}-\mu_0\right)^2\right) &\leq k \\
-\frac{n}{2\sigma^2}\left(\overline{\mathrm{X}}-\mu_0\right)^2 &\leq \log k \\
\frac{\left(\overline{\mathrm{X}}-\mu_0\right)^2}{\sigma^2/n} &\geq -2\log k \\
\frac{|\overline{\mathrm{X}}-\mu_0|}{\sigma/\sqrt{n}} &\geq \sqrt{-2\log k}.
\end{aligned}
$$

The type I error for this test is then equal to

$$
\begin{aligned}
\alpha &= P\left[\frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} \leq k\right] \\
\alpha &= P\left[\frac{|\overline{\mathrm{X}}-\mu_0|}{\sigma/\sqrt{n}} \geq \sqrt{-2\log k}\right]
\end{aligned}
$$

Since $\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$ is a normal random variable with mean 0 and variance 1, we then have

$$\alpha = P\left[|N(0,1)| \geq \sqrt{-2\log k}\right],$$

which means that

$$\begin{aligned}
\sqrt{-2\log k} &= z_{\alpha/2} \\
-2\log k &= (z_{\alpha/2})^2 \\
k &= \exp\left(-\frac{(z_{\alpha/2})^2}{2}\right),
\end{aligned}$$

which answers the question.

Remark 1. In particular, the critical region for this test is

$$\begin{aligned} C &= \left\{ (X_1, \ldots, X_n) \mid \frac{L(\widehat{\mu}_0)}{L(\widehat{\mu}_1)} \ \leq \ k \right\} \\ &= \left\{ (X_1, \ldots, X_n) \mid \frac{|\,\overline{\mathrm{X}} - \mu_0|}{\sigma/\sqrt{n}} \ \geq \ \sqrt{-2 \log k} \right\} \\ &= \left\{ (X_1, \ldots, X_n) \mid \frac{|\,\overline{\mathrm{X}} - \mu_0|}{\sigma/\sqrt{n}} \ \geq \ z_{\alpha/2} \right\}, \end{aligned}$$

which is the critical region that we have seen from Section 8.1.

11 Example: likelihood ratio test, Level 3

Let X be a normal random variable with unknown mean μ and **unknown variance** σ^2 (**notice the difference here**).

- The null hypothesis H_0 is $\mu = \mu_0,\ \sigma^2 \in [0, \infty)$.
- The alternative hypothesis H_0 is $\mu \neq \mu_0,\ \sigma^2 \in [0, \infty)$.

Compute the critical region for this likelihood ratio test with significance level α.

12 Answer: likelihood ratio test, Level 3

Recalll that the likelihood function for normal RV is (see what you wrote).

$$L(\mu, \sigma^2) \;=\; \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2\right).$$

For the null hypothesis H_0, $\mu = \mu_0$, $\sigma^2 \in [0, \infty)$.

The MLEs for this regime are (write this down)

$$\widehat{\mu}_0 = \mu_0; \qquad \widehat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2.$$

Therefore we get

$$\begin{aligned} L(\widehat{\mu}_0, \widehat{\sigma}_0^2) \;&=\; \left(\frac{1}{2\pi\,\widehat{\sigma}_0^2}\right)^{n/2} \exp\left(-\frac{1}{2\,\widehat{\sigma}_0^2}\sum_{i=1}^{n}(X_i-\mu_0)^2\right) \\ &=\; \left(\frac{1}{2\pi\,\widehat{\sigma}_0^2}\right)^{n/2} \exp\left(\frac{-n}{2}\right). \end{aligned}$$

For the alternative hypothesis H_1, $\mu \in (-\infty, \infty)$, $\sigma^2 \in [0, \infty)$. The MLEs for this regime is (write this down)

$$\widehat{\mu}_1 = \overline{\mathrm{X}}; \qquad \widehat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2.$$

(**Important**: Here we use the variance of the sample distribution, **not** the sample variance.)

Therefore we get

$$\begin{aligned} L(\widehat{\mu}_1, \widehat{\sigma}_1^2) &= \left(\frac{1}{2\pi\,\widehat{\sigma}_1^2}\right)^{n/2} \exp\left(-\frac{1}{2\,\widehat{\sigma}_1^2}\sum_{i=1}^{n}\left(X_i - \overline{\mathrm{X}}\right)^2\right) \\ &= \left(\frac{1}{2\pi\,\widehat{\sigma}_1^2}\right)^{n/2} \exp\left(\frac{-n}{2}\right). \end{aligned}$$

Therefore the ratio $\frac{L(\widehat{\mu}_0, \widehat{\sigma}_0^2)}{L(\widehat{\mu}_1, \widehat{\sigma}_1^2)}$ is (BT)

$$
\begin{aligned}
\frac{L(\widehat{\mu}_0, \widehat{\sigma}_0^2)}{L(\widehat{\mu}_1, \widehat{\sigma}_1^2)} &= \frac{\left(\frac{1}{2\pi\,\widehat{\sigma}_0^2}\right)^{n/2} \exp\left(\frac{-n}{2}\right)}{\left(\frac{1}{2\pi\,\widehat{\sigma}_1^2}\right)^{n/2} \exp\left(\frac{-n}{2}\right)} \\
&= \left(\frac{\widehat{\sigma}_1^2}{\widehat{\sigma}_0^2}\right)^{n/2} \\
&= \left(\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^2}\right)^{n/2} \\
&= \left(\frac{\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2}{\sum_{i=1}^{n}(X_i - \mu_0)^2}\right)^{n/2}.
\end{aligned}
$$

Now recall (see what you wrote)

$$\sum_{i=1}^{n}(X_i - \mu_0)^2 \;=\; \sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2 \quad + \quad n(\overline{\mathrm{X}} - \mu_0)^2,$$

so we have

$$\begin{aligned}
\frac{L(\widehat{\mu}_0, \widehat{\sigma}_0^2)}{L(\widehat{\mu}_1, \widehat{\sigma}_1^2)} &= \left(\frac{\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2}{\sum_{i=1}^{n}(X_i - \mu_0)^2}\right)^{n/2} \\
&= \left(\frac{\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2}{\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2 \quad + \quad n(\overline{\mathrm{X}} - \mu_0)^2}\right)^{n/2} \\
&= \left(\frac{1}{1 + \frac{n(\overline{\mathrm{X}} - \mu_0)^2}{\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2}}\right)^{n/2}.
\end{aligned}$$

Now note that, the likelihood ratio test is checking if

$$\frac{L(\widehat{\mu}_0, \widehat{\sigma}_0^2)}{L(\widehat{\mu}_1, \widehat{\sigma}_1^2)} \le k.$$

This is equivalent to (BT) (write this down)

$$\begin{aligned}
\left(\frac{1}{1+\frac{n(\overline{\mathrm{X}}-\mu_0)^2}{\sum_{i=1}^n (X_i-\overline{\mathrm{X}})^2}}\right)^{n/2} &\le k \\
\frac{1}{1+\frac{n(\overline{\mathrm{X}}-\mu_0)^2}{\sum_{i=1}^n (X_i-\overline{\mathrm{X}})^2}} &\le k^{2/n} \\
\frac{1}{k^{2/n}} &\le 1+\frac{n(\overline{\mathrm{X}}-\mu_0)^2}{\sum_{i=1}^n (X_i-\overline{\mathrm{X}})^2} \\
k^{-2/n}-1 &\le \frac{n(\overline{\mathrm{X}}-\mu_0)^2}{\sum_{i=1}^n (X_i-\overline{\mathrm{X}})^2} \\
(n-1)(k^{-2/n}-1) &\le \frac{n(\overline{\mathrm{X}}-\mu_0)^2}{\frac{1}{n-1}\sum_{i=1}^n (X_i-\overline{\mathrm{X}})^2}.
\end{aligned}$$

This is equal to

$$\frac{n(\overline{\mathrm{X}}-\mu_0)^2}{\frac{1}{n-1}\sum_{i=1}^n (X_i-\overline{\mathrm{X}})^2} \ge (n-1)(k^{-2/n}-1).$$

Now recall that the sample variance s is

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{\mathrm{X}})^2.$$

So our inequality becomes

$$\begin{aligned}\frac{n(\overline{\mathrm{X}}-\mu_0)^2}{s^2} &\geq (n-1)(k^{-2/n}-1)\\ \frac{\sqrt{n}\,|\,\overline{\mathrm{X}}-\mu_0|}{s} &\geq \sqrt{(n-1)(k^{-2/n}-1)}.\end{aligned}$$

Hence the type I error for this test is

$$\begin{aligned}\alpha &= P\left[\frac{L(\widehat{\mu}_0, \widehat{\sigma}_0^2)}{L(\widehat{\mu}_1, \widehat{\sigma}_1^2)} \leq k\right]\\ &= P\left[\frac{\sqrt{n}\,|\,\overline{\mathrm{X}}-\mu_0|}{s} \geq \sqrt{(n-1)(k^{-2/n}-1)}\right].\end{aligned}$$

Since X is a normal random variable, we have

$$T := \frac{\sqrt{n}(\overline{\mathrm{X}} - \mu_0)}{s}$$

is a t-student random variable with $n-1$ degrees of freedom. So we have

$$\alpha = P\left[|T| \ \geq \ \sqrt{(n-1)(k^{-2/n}-1)}\right],$$

which is equivalent to

$$\begin{aligned}
\sqrt{(n-1)(k^{-2/n}-1)} &= t_{\alpha/2}(n-1) \\
(n-1)(k^{-2/n}-1) &= (t_{\alpha/2}(n-1))^2 \\
k^{-2/n} &= \frac{(t_{\alpha/2}(n-1))^2}{n-1} + 1 \\
k &= \left[\frac{(t_{\alpha/2}(n-1))^2}{n-1} + 1\right]^{-n/2},
\end{aligned}$$

which is our answer.

Remark 2. In particular, the critical region for this test is

$$\begin{aligned} C &= \left\{ (X_1, \ldots, X_n) \mid \frac{L(\widehat{\mu}_0, \widehat{\sigma}_0^2)}{L(\widehat{\mu}_1, \widehat{\sigma}_1^2)} \ \le \ k \right\} \\ &= \left\{ (X_1, \ldots, X_n) \mid \frac{\sqrt{n}\,|\,\overline{\mathrm{X}} - \mu_0|}{s} \ \ge \ \sqrt{(n-1)(k^{-2/n} - 1)} \right\} \\ &= \left\{ (X_1, \ldots, X_n) \mid \frac{\sqrt{n}\,|\,\overline{\mathrm{X}} - \mu_0|}{s} \ \ge \ t_{\alpha/2}(n-1) \right\}, \end{aligned}$$

which is the critical region that we have seen from Section 8.2.