Math 170S

Lecture Notes Section 8.7 [*][†]

Best critical region

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on Hanbaek Lyu's and Liza Rebrova's notes from the previous quarter, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Example: Chocolate, Level 1

Let X be the sugar content of a random chocolate bar, which is a normal random variable with unknown mean μ and variance 36.

- The company claims that $\mu = 50$ (the null hypothesis H_0);
- However, the Federal Trade Commission claims that $\mu = 55$ (the alternative hypothesis H_1).

Let $X_1, \ldots, X_n$ be $n = 16$ random samples for X. Your friendly instructor wants to find a test so that the type I error α is equal to 0.05.

Suppose that there are two choices of critical regions:

- The critical region C,

  $$C = \{(X_1, \ldots, X_n) \mid X_1 + X_2 + \ldots + X_n \geq 839.52\}.$$

  This is our usual choice of critical region.

- The critical region D,

  $$D = \{(X_1, \ldots, X_n) \mid X_1 + 2X_2 + \ldots + nX_n \geq 7183\}.$$

Both critical regions have type I error $\alpha \approx 0.05$:

- For the critical region C,

$$\begin{aligned} \alpha &= P[\text{reject } H_0 \text{ given that } H_0 \text{ is true}] \\ &= P[X_1 + X_2 + \ldots + X_n \geq 839.52 \text{ given that } \mu = 50]. \end{aligned}$$

Note that $X_1 + \ldots + X_n$ is a normal RV with mean $n\mu_0 = (16)(50)$ and variance $n\sigma^2 = (16)(36)$, so

$$\begin{aligned} \alpha =& 1 - \Phi\left(\frac{(839.52) - (16)(50)}{\sqrt{(16)(36)}}\right) \\ =& 1 - \Phi(1.647) \approx 0.05. \end{aligned}$$

- For the critical region D,

$$\begin{aligned}\alpha &= P[\text{reject } H_0 \text{ given that } H_0 \text{ is true}] \\ &= P\left[X_1 + 2X_2 + \ldots + nX_n \geq 7183 \text{ given } \mu = 50\right].\end{aligned}$$

Note $X_1 + 2X_2 + \ldots + nX_n$ is a normal RV with

$$\begin{aligned}\text{mean} &= E[X_1] + 2E[X_2] + \ldots + nE[X_n] \\ &= \frac{n(n+1)}{2}\mu_0 \;=\; 6800; \\ \text{variance} &= \text{var}[X_1] + (2)^2\text{var}[X_2] + \ldots + (n)^2\text{var}[X_n] \\ &= \frac{n(n+1)(2n+1)}{6}\sigma^2 = 53856,\end{aligned}$$

so we have

$$\begin{aligned}\alpha =& 1 - \Phi\left(\frac{(7183) - (6800)}{\sqrt{53856}}\right) \\ =& 1 - \Phi(1.65) \;\approx\; 0.05.\end{aligned}$$

Both C and D have comparable type I error, so we choose the one with smaller the type II error β.

- For critical region C,

$$\begin{aligned}\beta =& P[\text{not rejecting } H_0 \text{ given that } H_1 \text{ is true}] \\ =& P[X_1 + \ldots + X_n < 839.52 \text{ given that } \mu = 55].\end{aligned}$$

  Recall that $X_1 + \ldots + X_n$ is a normal RV with mean $n\mu_1 = (16)(55)$ and variance $n\sigma^2 = (16)(36)$, so

$$\begin{aligned}\beta =& \Phi\left(\frac{(839.52) - (16)(55)}{\sqrt{(16)(36)}}\right) \\ =& \Phi(-1.69) \;\approx\; 0.0455.\end{aligned}$$

- For the critical region D,

$$\beta = P[\text{not rejecting } H_0 \text{ given that } H_1 \text{ is true}]$$
$$= P\left[X_1 + 2X_2 + \ldots + nX_n < 7183 \text{ given } \mu = 55\right].$$

Recall $X_1 + 2X_2 + \ldots + nX_n$ is a normal RV with

$$\begin{aligned} \text{mean} &= \frac{n(n+1)}{2}\mu_1 = 7480; \\ \text{variance} &= \frac{n(n+1)(2n+1)}{6}\sigma^2 = 53856. \end{aligned}$$

$$\begin{aligned} \beta =& \Phi\left(\frac{(7183) - (7480)}{\sqrt{53856}}\right) \\ =& \Phi(-1.28) \approx 0.1003. \end{aligned}$$

So the region C has smaller type II error, and hence a better critical region. We will see that C is actually the best critical region for significance level $\alpha = 0.05$.

2 Setting: best critical region

Object: X is a random variable with density f_θ with **unknown** θ.

Hypotheses:

- **Null Hypothesis** H_0: θ is equal to θ_0.
- **Alternative Hypothesis** H_1: θ is equal to θ_1.

Input: Random samples $X_1, \ldots, X_n$ for X and significance level α.

Methodology:

Reject H_0 if $(X_1, \ldots, X_n)$ is contained in the (to be determined) critical region C, and do not reject H_0 otherwise.

Problem:

- Find the **best critical region** C with significance level (type I error) α. The test that uses this C is called a **uniformly most powerful test**.

3 Definition: best critical region

Definition 1. C is the *best critical region of size* α if the type II error is the smallest among all critical regions with significance level α.

4 Theorem: Neyman-Pearson

Recall the definition of the maximum likelihood function

$$L(\theta) = f_\theta(x_1) \dots f_\theta(x_n).$$

Remember that θ is **unknown** and $x_1, \dots, x_n$ are **variables**.

Theorem 2 (Neyman-Pearson lemma)**.** *A critical region C of size α is the best critical region if there exists k such that (write this down)*

- *The type I error $P[(X_1, \dots, X_n \in C); \theta = \theta_0]$ is equal to α;*
- *$\frac{L(\theta_0)}{L(\theta_1)} \leq k$ for all $(x_1, \dots, x_n)$ in C; and*
- *$\frac{L(\theta_0)}{L(\theta_1)} \geq k$ for all $(x_1, \dots, x_n)$ outside of C.*

5 Example: Chocolate, Level 2

Recall the chocolate example, X is a random variable with unknown mean μ and variance 36. We have $n = 16$ random sample $X_1, \ldots X_n$ for X.

- The null hypothesis is $\mu = \mu_0 = 50$;
- The alternative hypothesis $\mu = \mu_1 = 55$.

We will show that the critical region

$$C = \{(X_1, \ldots, X_n) \mid X_1 + X_2 + \ldots + X_n \geq 839.52\}$$

is the best critical region of size 0.05.

6 Answer: Chocolate, Level 2

We first compute the maximum likelihood function $L(\mu)$.

We have (BT)

$$\begin{aligned} L(\mu) &= f_\mu(x_1)\dots f_\mu(x_n) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x_i-\mu)^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right) \\ &= (72\pi)^{-n/2} \exp\left(-\frac{1}{72}\sum_{i=1}^{n}(x_i-\mu)^2\right). \end{aligned}$$

The ratio $\frac{L(\mu_0)}{L(\mu_1)}$ is then equal to (BT)

$$\begin{aligned}
\frac{L(50)}{L(55)} &= \frac{(72\pi)^{-n/2}\exp\left(-\frac{1}{72}\sum_{i=1}^{n}(x_i-50)^2\right)}{(72\pi)^{-n/2}\exp\left(-\frac{1}{72}\sum_{i=1}^{n}(x_i-55)^2\right)} \\
&= \exp\left(\frac{-1}{72}\sum_{i=1}^{n}\left[(x_i-50)^2-(x_i-55)^2\right]\right) \\
&= \exp\left(\frac{-1}{72}\sum_{i=1}^{n}\left[-2x_i(50-55)+(50^2-55^2)\right]\right) \\
&= \exp\left(\frac{-1}{72}\sum_{i=1}^{n}[10x_i-525]\right) \\
&= \exp\left(\frac{-10(x_1+\ldots+x_n)+(16)(525)}{72}\right).
\end{aligned}$$

We now check the three conditions in Neyman-Pearson lemma:

- For the first condition,

$$
\begin{aligned}
& P[(X_1, \ldots, X_n \in C); \mu = \mu_0] \\
&= P[X_1 + X_2 + \ldots + X_n \geq 839.52 \text{ given that } \mu = 50] \\
&= 1 - \Phi\left(\frac{(839.52) - n\mu_0}{\sqrt{n\sigma^2}}\right) \\
&= 1 - \Phi\left(\frac{(839.52) - (16)(50)}{\sqrt{(16)(36)}}\right) \\
&= 1 - \Phi(1.647) \;\approx\; 0.05.
\end{aligned}
$$

- For the second condition, We choose k to be

$$k := \exp\left(\frac{-10(839.52) + (16)(525)}{72}\right).$$

Now note that

$$(x_1, \ldots, x_n) \text{ in } C \quad \text{means that}$$
$$x_1 + \ldots + x_n \geq 839.52.$$

So we have

$$\begin{aligned}
\frac{L(50)}{L(55)} &= \exp\left(\frac{-10(x_1 + \ldots + x_n) + (16)(525)}{72}\right) \\
&\leq \exp\left(\frac{-10(839.52) + (16)(525)}{72}\right) \\
&= k.
\end{aligned}$$

- For the third condition,

$$(x_1, \ldots, x_n) \text{ outside of } C \quad \text{means that}$$

$$x_1 + \ldots + x_n < 839.52.$$

So we have

$$\begin{aligned} \frac{L(50)}{L(55)} &= \exp\left(\frac{-10(x_1 + \ldots + x_n) + (16)(525)}{72}\right) \\ &> \exp\left(\frac{-10(839.52) + (16)(525)}{72}\right) \\ &= k\,. \end{aligned}$$

Hence we conclude that C is indeed the best critical region of size 0.05.

7 Example: Chocolate, Level 3

Let X be a normal RV with unknown mean μ and variance 36. We have n random sample $X_1, \ldots X_n$ for X.

- The null hypothesis is $\mu = \mu_{\mathbf{0}}$;
- The alternative hypothesis $\mu = \mu_1$.

Assume that $\mu_0 < \mu_1$. Find a best critical region C of size α.

8 Answer: Chocolate, Level 3

Recall the maximum likelihood function:

$$L(\mu) \;=\; = (72\pi)^{-n/2} \exp\left(-\frac{1}{72}\sum_{i=1}^{n}(x_i-\mu)^2\right).$$

We can then compute the ratio $\frac{L(\mu_0)}{L(\mu_1)}$ by (BT)

$$\begin{aligned}
\frac{L(\mu_0)}{L(\mu_1)} &= \frac{(72\pi)^{-n/2}\exp\left(-\frac{1}{72}\sum_{i=1}^{n}(x_i-\mu_0)^2\right)}{(72\pi)^{-n/2}\exp\left(-\frac{1}{72}\sum_{i=1}^{n}(x_i-\mu_1)^2\right)} \\
&= \exp\left(\frac{-1}{72}\sum_{i=1}^{n}\left[-2x_i(\mu_0-\mu_1)+(\mu_0^2-\mu_1^2)\right]\right) \\
&= \exp\left(\frac{2(\mu_0-\mu_1)(\sum_{i=1}^{n}x_i)-n(\mu_0^2-\mu_1^2)}{72}\right) \\
&= \exp\left(\frac{2n(\mu_0-\mu_1)\,\overline{\mathrm{x}}\,-n(\mu_0^2-\mu_1^2)}{72}\right).
\end{aligned}$$

For C to be a best critical region, we need to find k so that

- (2nd cond.) For all $(x_1, \ldots, x_n)$ in C, we need (BT)

$$\begin{aligned} \frac{L(\mu_0)}{L(\mu_1)} &\le k \\ \exp\left(\frac{2n(\mu_0-\mu_1)\,\overline{\mathrm{x}} - n(\mu_0^2-\mu_1^2)}{72}\right) &\le k \\ \frac{2n(\mu_0-\mu_1)\,\overline{\mathrm{x}} - n(\mu_0^2-\mu_1^2)}{72} &\le \log k. \end{aligned}$$

Continuing the calculation,

$$\begin{aligned} 2n(\mu_0-\mu_1)\,\overline{\mathrm{x}} &\le 72\log k + n(\mu_0^2-\mu_1^2) \\ \overline{\mathrm{x}} &\ge \frac{72\log k + n(\mu_0^2-\mu_1^2)}{2n(\mu_0-\mu_1)} \\ \overline{\mathrm{x}} &\ge \frac{72\log k}{2n(\mu_0-\mu_1)} + \frac{\mu_0+\mu_1}{2}. \end{aligned}$$

- (3rd cond.) For all $(x_1, \ldots, x_n)$ outside of C, we need (ST)

$$\begin{aligned} \frac{L(\mu_0)}{L(\mu_1)} &\geq k \\ \exp\left(\frac{2n(\mu_0 - \mu_1)\,\bar{\mathrm{x}} - n(\mu_0^2 - \mu_1^2)}{72}\right) &\geq k. \end{aligned}$$

$$\begin{aligned} \frac{2n(\mu_0 - \mu_1)\,\bar{\mathrm{x}} - n(\mu_0^2 - \mu_1^2)}{72} &\geq \log k \\ 2n(\mu_0 - \mu_1)\,\bar{\mathrm{x}} &\geq 72\log k + n(\mu_0^2 - \mu_1^2) \\ \bar{\mathrm{x}} &\leq \frac{72\log k + n(\mu_0^2 - \mu_1^2)}{2n(\mu_0 - \mu_1)} \\ \bar{\mathrm{x}} &\leq \frac{72\log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}. \end{aligned}$$

Remark 3. A shorter way to do this: “By the same argument as before, we conclude that ...”.

So we conclude

- For $(x_1, \ldots, x_n)$ in C, we need

$$\overline{\mathrm{x}} \;\geq\; \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}.$$

- For $(x_1, \ldots, x_n)$ outside of C, we need

$$\overline{\mathrm{x}} \;\leq\; \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}.$$

Thus the best critical region C is of the form (write this down)

$$C = \left\{ (X_1, \ldots, X_n) \;\mid\; \overline{\mathrm{X}} \geq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2} \right\},$$

where k is to be determined.

We can determine k with the 1st cond.,

$$\begin{aligned} P[(X_1, \ldots, X_n) \in C; \mu = \mu_0] &= \alpha \\ P\left[\overline{\mathrm{X}} \geq \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}; \ \mu = \mu_0\right] &= \alpha. \end{aligned}$$

We write the term in the probability above as

$$\text{Scary} \ := \ \frac{72 \log k}{2n(\mu_0 - \mu_1)} + \frac{\mu_0 + \mu_1}{2}$$

Since $\overline{\mathrm{X}}$ is normal RV with mean μ_0 and variance $\frac{\sigma^2}{n} = \frac{36}{n}$,

(write this down)

$$\begin{aligned}
P[(X_1, \ldots, X_n) \in C; \mu = \mu_0] &= \alpha \\
P\left[\overline{\mathrm{X}} \geq \text{Scary};\ \mu = \mu_0\right] &= \alpha \\
1 - \Phi\left(\frac{\text{Scary} - \mu_0}{6/\sqrt{n}}\right) &= \alpha \\
\frac{\text{Scary} - \mu_0}{6/\sqrt{n}} &= \Phi^{-1}(1-\alpha) \\
\text{Scary} - \mu_0 &= (\frac{6}{\sqrt{n}})\Phi^{-1}(1-\alpha) \\
\text{Scary} &= \mu_0 + (\frac{6}{\sqrt{n}})\Phi^{-1}(1-\alpha).
\end{aligned}$$

Substituting the last equation into the formula for critical region, we get

$$C = \left\{ (X_1, \ldots, X_n) \ \mid \ \overline{\mathrm{X}} \geq \mu_0 + (\frac{6}{\sqrt{n}})\Phi^{-1}(1-\alpha) \right\},$$

which is our answer. (Write this down.)

9 Re-example: Chocolate, Level 2

Let X be a normal random variable with unknown mean μ and variance 36. We have $n = 16$ random sample $X_1, \ldots X_n$ for X.

- The null hypothesis is $\mu = 50$;
- The alternative hypothesis $\mu = 55$.

Find a best critical region C of size 0.05.

10 Re-answer: Chocolate, Level 2

Here we have

$$n = 16; \qquad \mu_0 = 50; \qquad \alpha = 0.05.$$

So we have

$$\begin{aligned} \mu_0 + (\frac{6}{\sqrt{n}})\Phi^{-1}(1-\alpha) &= (50) + (\frac{6}{\sqrt{16}})\Phi^{-1}(1-0.05) \\ &\approx 52.47, \end{aligned}$$

and therefore the critical region is

$$C = \left\{(X_1, \ldots, X_n) \;\mid\; \overline{\mathrm{X}} \geq 52.47\right\},$$

which is equivalent to the critical region from Level 2.

11 Example: Chocolate, Level 4

Let X be a normal random variable with unknown mean μ and variance 36. We have n random sample $X_1, \ldots X_n$.

- The null hypothesis is $\mu = 50$;
- The **alternative hypothesis** $\mu > 50$.

Find a best critical region C of size 0.05.

12 Answer: Chocolate, Level 4

The key observation here is that the best critical region C from Level 3

$$C = \left\{ (X_1, \ldots, X_n) \ \mid \ \overline{\mathrm{X}} \geq \mu_0 + (\frac{6}{\sqrt{n}})\Phi^{-1}(1-\alpha) \right\},$$

does not depend on the mean μ_1 from the alternative hypothesis. We can therefore use C for the best critical region for all $\mu > 50$.

By the same calculation as in Level 2, we conclude that

$$C = \left\{ (X_1, \ldots, X_n) \ \mid \ \overline{\mathrm{X}} \geq 52.47 \right\}.$$

Remark 4. Recall the definition of sufficient statistics $u := u(x_1, \dots, x_n)$ from Section 6. If a sufficient statistics u exists, then the ratio $\frac{L(\theta_0)}{L(\theta_1)}$ is

$$\frac{L(\theta_0)}{L(\theta_1)} = \frac{\phi(u, \theta_0)\, h(x_1, \dots, x_n)}{\phi(u, \theta_1)\, h(x_1, \dots, x_n)} = \frac{\phi(u, \theta_0)}{\phi(u, \theta_1)}.$$

By the Neyman-Pearson lemma, this means that the best critical region are usually based on the sufficient statistics when they exist, e.g.,

$$\begin{aligned} C =& \{(x_1, \dots, x_n) \mid u(x_1, \dots, x_n) \geq 42\}, \quad \text{or} \\ C =& \{(x_1, \dots, x_n) \mid |u(x_1, \dots, x_n) - 2| \geq 67\}. \end{aligned}$$