Math 170S

Lecture Notes Section 8.6 [*][†]

Power function

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on Hanbaek Lyu's and Liza Rebrova's notes from the previous quarter, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Example: Type I and type II error (recap)

A detective is investigating if the friendly instructor likes cheesy romance movies. Let X be the hatred indicator for a random romance movie.

- X is equal to 0 if the friendly instructor likes the romance movie;
- X is equal to 1 if the friendly instructor hates the romance movie.

The expert team told you that X is Bernoulli RV with parameter p

- If the friendly instructor is innocent (the null hypothesis H_0), then $p = 1/2$.
- If the friendly instructor is guilty (the alternative hypothesis H_1), then $p = 1/4$.

You then hacked into the friendly instructor's Netflix account and get the sample values $X_1, \ldots, X_{20}$ for X from 20 romance movies he watched.

- If $\overline{\mathrm{X}} \leq 0.3$, then the friendly instructor is guilty and he will be roasted by the whole class for his interest;
- If $\overline{\mathrm{X}} > 0.3$, then no actions will be taken against the friendly instructor.

We are interested in two questions:

- What is the probability that you make a false accusation against the friendly instructor? (i.e., reject H_0 when H_0 is true).
- What is the probability that the friendly instructor gets away scot-free from his crime? (i.e., do not reject H_0 when H_0 is false).

2 Definition: Type I and Type II error (recap)

Type I error, denoted by α, is the probability of rejecting H_0 when H_0 is true, sometimes also called statistical significance.

Type II error, denoted by β, is the probability of not rejecting H_0 when H_0 is false.

It is usually thought that Type I error is more important, as we do not want to falsely accuse people of a crime. However, in this section we will care about both Type I and Type II error.

3 Answer: Type I and type II error (recap)

We first compute α. Let Y be the random variable

$$Y := X_1 + X_2 + \ldots + X_{20} = n\,\overline{\mathrm{X}}\,.$$

Note that

- We reject H_0 if $Y \leq 20(0.3) = 6$, and we do not reject H_0 if $Y > 6$.
- Y is the binomial random variable with parameter p and $n = 20$.

We can then compute the Type I error α by

$$
\begin{aligned}
\alpha =& P[\text{Reject } H_0 \mid H_0 \text{ is true}] = P\left[Y \leq 6 \mid p = \frac{1}{2}\right] \\
=& \sum_{y=0}^{6} \binom{20}{y} \left(\frac{1}{2}\right)^{y} \left(\frac{1}{2}\right)^{20-y} \;=\; 0.0577.
\end{aligned}
$$

We now compute the type II error β by

$$
\begin{aligned}
\beta &= P[\text{Not rejecting } H_0 \mid H_0 \text{ is false}] \\
&= 1 - P[\text{rejecting } H_0 \mid H_0 \text{ is false}] \\
&= 1 - P\left[Y \leq 6 \mid p = \frac{1}{4}\right] \\
&= 1 - \sum_{y=0}^{6} \binom{20}{y} \left(\frac{1}{4}\right)^{y} \left(\frac{3}{4}\right)^{20-y} \;=\; 0.2142.
\end{aligned}
$$

4 Example: Type I and Type II error, composite alternative hypothesis

Consider the Netflix example, but with the alternative hypothesis changed,

- The null hypothesis H_0: X is a Bernoulli random variable with $p = 1/2$.
- The alternative hypothesis H_1: X is a Bernoulli random variable with $\mathbf{p < 1/2}$.

Compute the type I error α and type II error β.

5 Answer: Type I and Type II error, composite alternative hypothesis

Type I error α is the same as before.

$$\begin{aligned}\alpha =& P[\text{Reject } H_0 \mid H_0 \text{ is true}] = P\left[Y \le 6 \mid p = \frac{1}{2}\right] \\ =& \sum_{y=0}^{6} \binom{20}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{20-y} \quad = \quad 0.0577.\end{aligned}$$

However, β now depends on the unknown parameter p:

$$\begin{aligned}\beta =& 1 - P[\text{rejecting } H_0 \mid H_0 \text{ is false}] \\ =& 1 - P\left[Y \le 6 \textbf{ given the unknown } p\right] \\ =& 1 - \sum_{y=0}^{6} \binom{20}{y} (p)^y (1-p)^{20-y},\end{aligned}$$

which is a function of p.

6 Definition: power function

Let X be a random variable with unknown parameter μ.

- Null hypothesis H_0: $\mu = \mu_0$;
- Alternative hypothesis H_1.

Definition 1. The **power function** $K(\mu)$ is

$$K(\mu) := P[\text{rejecting } H_0 \text{ given the unknown } \mu].$$

$K(\mu)$ depends on **the rule of rejecting H_0**.

Remark 2. Some other textbooks define the power function as

$$K(\mu) := 1 - P[\text{rejecting } H_0 \text{ given the unknown } \mu].$$

We will follow the definition in our textbook and uses the formula in Definition 1.

7 Theorem: power function

Theorem 3. *The type I error* α *and the type II error* β *are given by*

$$\alpha \;=\; K(\mu_0); \qquad \beta \;=\; 1 - K(\mu).$$

The power function can be used to both determine β and to determine the critical region.

8 Example: power function

Let X be a **chi-square random variable** with **unknown** r degrees of freedom. Suppose that:

- The null hypothesis H_0 is $r = r_0 = 1$;
- The alternative hypothesis H_1 is $r > 1$.

Let X_1, X_2, X_3, X_4 be 4 random samples for X.

1. Compute the critical region for the sample mean $\overline{\mathrm{X}}$ with significance level 0.025.
2. Compute the type II error β given that $r = 5$.

9 Answer: chi-square random variable

We are in case (a), so the critical region is of the form

$$\big[r_0 + c, \infty\big) \ = \ \big[1 + c, \infty\big),$$

for some error c that we want to compute. We now compute the power function $K(r)$,

$$\begin{aligned} K(r) \ &= P[\text{rejecting } H_0 \text{ given the unknown } r] \\ &= P[\overline{\mathrm{X}} \geq 1 + c \quad \text{given } r] \\ &= P[X_1 + X_2 + X_3 + X_4 \geq 4(1 + c) \quad \text{given } r]. \end{aligned}$$

Now note that $X_1+X_2+X_3+X_4$ is a chi-square random variable with $4r$ degrees of freedom, so

$$K(r) \ = P[\chi^2(4r) \geq 4(1+c)].$$

We can compute $K(r)$ using Table IV in textbook Appendix B. For the type I error α,

$$\alpha \ = \ K(r_0) \ = \ P[\chi^2(4) \geq 4(1+c)].$$

To get significance level 0.025, we need $\alpha = 0.025$, so

$$P[\chi^2(4) \geq 4(1+c)] \ = \ 0.025.$$

By the power of Table IV, we have $4(1+c) = 11.14$, so $c = 1.785$, and the critical region is

$$\big[2.785, \infty\big).$$

We now compute the type II error β when $r = 5$. We have

$$\beta = 1 - K(5) = 1 - P[\chi^2(20) \geq 11.14] \approx 1 - 0.943 = 0.057,$$

as desired.

10 Example: normal RV

Let X be a **normal random variable** with unknown mean μ and variance 100. Suppose that:

- The null hypothesis H_0 is $\mu = \mu_0 = 60$;
- The alternative hypothesis H_1 is $\mu > 60$.

Find the number of samples n so that

1. The type I error α is equal to 0.025; and
2. The type II error β, given that $\mu = 65$, is equal to 0.05.

11 Answer: normal RV

We are in case (a), so the critical region is of the form

$$\big[\mu_0 + c, \infty\big) \; = \; \big[60 + c, \infty\big),$$

for some error c that we want to compute. We now compute the power function $K(\mu)$,

$$\begin{aligned} K(\mu) \; &= P[\text{rejecting } H_0 \text{ given the unknown } \mu] \\ &= P[\overline{\mathrm{X}} \geq 60 + c \quad \text{given the unknown } \mu]. \end{aligned}$$

Since $\overline{\mathrm{X}}$ is a normal RV with mean μ and variance $\frac{100}{n}$, so

$$K(\mu) \; = 1 - \Phi\left(\frac{60 + c - \mu}{10/\sqrt{n}}\right),$$

where Φ is the cdf of the standard normal RV.

We now compute the type I error by using the power function. We have

$$\alpha \;=\; K(60) \;=\; 1 - \Phi\left(\frac{c}{10/\sqrt{n}}\right).$$

We now compute the type II error β when $\mu = 65$. We have

$$\beta \;=\; 1 - K(65) \;=\; \Phi\left(\frac{c-5}{10/\sqrt{n}}\right).$$

Plugging $\alpha = 0.025$ and $\beta = 0.05$ from the question,

$$1 - \Phi\left(\frac{c}{10/\sqrt{n}}\right) = 0.025 \quad \text{and} \quad \Phi\left(\frac{c-5}{10/\sqrt{n}}\right) = 0.05.$$

By the power of the magical table, we then have

$$\begin{aligned} \frac{c}{10/\sqrt{n}} &= \Phi^{-1}(1-0.025) = 1.96 \\ \frac{c-5}{10/\sqrt{n}} &= \Phi^{-1}(0.05) \ = \ -1.645. \end{aligned}$$

Solving these inequalities simultaneously, we obtain

$$\begin{aligned} 1.96 - \frac{5}{10/\sqrt{n}} \ &= \ -1.645 \\ \frac{10}{\sqrt{n}} \ &= \frac{5}{3.605} \\ n \ &\approx 51.98, \end{aligned}$$

so we will take $n = 52$ as our answer.

12 Example: Bernoulli RV

Let X be a **Bernoulli random variable** with unknown parameter p.

- The null hypothesis H_0 is $p = p_0 = \frac{1}{2}$;
- The alternative hypothesis H_1 is $p < \frac{1}{2}$.

We have $n = 31$ samples.

- Reject the null hypothesis H_0 if $X_1 + \ldots + X_{31} \leq 11$;
- Test is inconclusive if $X_1 + \ldots + X_{31} > 11$.

Approximate the type II error β if $p = \frac{1}{4}$.

13 Answer: Bernoulli RV

The power function $K(p)$ is given by

$$K(p) \;=\; P[X_1 + \ldots + X_{31} \leq 11 \quad \text{given unknown } p].$$

Let Y be the random variable $X_1 + \ldots + X_{31}$. We approximate Y by the normal RV with mean and variance

$$E[Y] = np; \qquad \text{Var}[Y] = np(1-p),$$

so we have

$$\begin{aligned} K(p) &= P[Y \leq 11 \quad \text{given unknown } p] \\ &= \Phi\left(\frac{11 + 0.5 - np}{\sqrt{np(1-p)}}\right). \end{aligned}$$

Plugging $n = 31$ and $p = \frac{1}{4}$, we have

$$K(\frac{1}{4}) \;=\; \Phi\left(\frac{11 + 0.5 - (31)(\frac{1}{4})}{\sqrt{(31)(\frac{3}{4})(\frac{1}{4})}}\right) \;=\; \Phi(1.28) = 0.8997,$$

so the Type II error is

$$\beta \;=\; 1 - K(\frac{1}{4}) \;=\; 0.1003,$$

as desired.