Math 170S

Lecture Notes Section 8.5 [*][†]

Tests about median

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on Hanbaek Lyu's and Liza Rebrova's notes from the previous quarter, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Example: median, one variable

The friendly instructor's sister has been very worried that her dog is gaining weight from its couch-potato lifestyle. The following is $n = 10$ samples values for the weight X of her dog (ordered from smallest to largest):

$$2.0 \quad 2.6 \quad 2.8 \quad 3.9 \quad 4.3 \quad 5.0 \quad 5.0 \quad 5.8 \quad 6.1 \quad 6.4.$$

Before this change of lifestyle, the **median** of the dog's weight is 3.7.

Assume that X is a **continuous symmetric** RV.

Can we conclude that the dog has been gaining weight with significance level $\alpha = 0.05$?

2 Setting: median, one variable

Object: X is an **continuous symmetric** random variables with **unknown median** m.

Hypotheses:

- **Null Hypothesis** H_0: The median m is equal to m_0.
- **Alternative Hypothesis** H_1: One of these three forms:
  (a) m is **strictly greater than** m_0;
  (b) m is **strictly smaller than** m_0;
  (c) m is **not equal to** m_0.

Input: Random samples $x_1, \ldots, x_n$ for X and significance level α.

Methodology:

- Compute the **Wilcoxon's signed rank statistic** W;
- Compute the critical region that depends on α; and
- Compute the p-value that depends on W.

Output:

- Reject the null hypothesis if W is contained in the critical region. Equivalently, reject the null hypothesis if the p-value is smaller than α.
- Do not reject the null hypothesis (i.e., test is inconclusive) otherwise.

3 Answer: median, one variable

We have $m_0 = 3.7$. The Wilcoxon signed rank statistics is computed as follows:

1. Compute the difference $x_i - m_0$:

x_i :	2.0	2.6	2.8	3.9	4.3	5.0	5.0	5.8	6.1	6.4
$x_i - m_0$:	−1.7	−1.1	−0.9	0.2	0.6	1.3	1.3	2.1	2.4	2.7.

2. Take the absolute value $|x_i - m_0|$:

x_i :	2.0	2.6	2.8	3.9	4.3	5.0	5.0	5.8	6.1	6.4
$x_i - m_0$:	−1.7	−1.1	−0.9	0.2	0.6	1.3	1.3	2.1	2.4	2.7
$\|x_i - m_0\|$:	1.7	1.1	0.9	0.2	0.6	1.3	1.3	2.1	2.4	2.7.

3. Rank the samples from smallest to largest based on the value of $|x_i - m_0|$:

x_i :	2.0	2.6	2.8	3.9	4.3	5.0	5.0	5.8	6.1	6.4
$x_i - m_0$:	−1.7	−1.1	−0.9	0.2	0.6	1.3	1.3	2.1	2.4	2.7
$\|x_i - m_0\|$:	1.7	1.1	0.9	0.2	0.6	1.3	1.3	2.1	2.4	2.7
Ranks:	7	4	3	1	2	5.5	5.5	8	9	10.

4. Put plus to the ranks if $x_i - m_0$ is positive, and put minus to the ranks if $x_i - m_0$ is negative:

x_i :	2.0	2.6	2.8	3.9	4.3	5.0	5.0	5.8	6.1	6.4
$x_i - m_0$:	−1.7	−1.1	−0.9	0.2	0.6	1.3	1.3	2.1	2.4	2.7
$\|x_i - m_0\|$:	1.7	1.1	0.9	0.2	0.6	1.3	1.3	2.1	2.4	2.7
Signed ranks:	−7	−4	−3	1	2	5.5	5.5	8	9	10.

5. Take the sum of the signed ranks of the samples. This is the **Wilcoxon signed rank statistic**:

$$W = (-7)+(-4)+(-3)+(1)+(2)+(5.5)+(5.5)+(8)+(9)+(10) = 27.$$

On the other hand, W is known to have expectation and variance

$$E[W] = 0; \qquad \mathrm{Var}[W] = \frac{n(n+1)(2n+1)}{6}.$$

In our case, this is

$$E[W] = 0; \qquad \mathrm{Var}[W] = \frac{(10)(11)(21)}{6} = 385.$$

We now approximate W by normal random variable,

$$W \sim N(0, 385).$$

We now compute the p-value of this test. The alternative hypothesis is that the dog is **gaining weight**. So

p-value = probability that W is **greater than or equal to** 27.

Since $W \sim N(0, 385)$ and the sample value for W is 27,

$$\begin{aligned} p\text{-value} &= P[W \geq 27] \;=\; 1 - \Phi\left(\frac{27 - E[W]}{\sqrt{\text{Var}[W]}}\right) \\ &= 1 - \Phi\left(\frac{27 - 0}{\sqrt{385}}\right) \;=\; 1 - \Phi(1.38) \\ &= 1 - (0.9162) \;=\; 0.0838. \end{aligned}$$

This p-value is not smaller than $\alpha = 0.05$, so this test is inconclusive.

4 Wilcoxon's signed rank statistic

Definition 1. Wilcoxon's signed rank statistic is

$$W := \sum_{i=1}^{n} \operatorname{sign}(x_i - m_0) \operatorname{Rank}(|x_i - m_0|),$$

where

- $\operatorname{sign}(x_i - m_0)$ is equal to $+1$ if $x_i - m_0$ is nonnegative, and is equal to -1 if $x_i - m_0$ is negative.
- $\operatorname{Rank}(|x_i - m_0|)$ is the rank of $|x_i - m_0|$ among the n samples, with 1 being the smallest rank and n being the largest rank. Ties receive a rank equal to the average of the ranks they span.

5 Theorem: median, one variable

Theorem 2. *(a) For the case $m > m_0$,*

$$\textit{critical region} \;=\; \left[z_\alpha\sqrt{\frac{n(n+1)(2n+1)}{6}}\;,\;\infty\right),$$
$$\textit{p-value} \;=\; 1-\Phi\left(\frac{W}{\sqrt{n(n+1)(2n+1)/6}}\right).$$

(b) For the case $m < m_0$,

$$\textit{critical region} \;=\; \left(-\infty\;,\;-z_\alpha\sqrt{\frac{n(n+1)(2n+1)}{6}}\right],$$
$$\textit{p-value} \;=\; \Phi\left(\frac{W}{\sqrt{n(n+1)(2n+1)/6}}\right).$$

(c) For the case $m \neq m_0$, we have

$$\begin{aligned}
\textit{critical region} &= \left(-\infty\ ,\ -z_{\alpha/2}\sqrt{\frac{n(n+1)(2n+1)}{6}} \right] \cup \\
&\quad \left[z_{\alpha/2}\sqrt{\frac{n(n+1)(2n+1)}{6}}\ ,\ \infty \right), \\
\textit{p-value} &= 2\left[1 - \Phi\left(\frac{|W|}{\sqrt{n(n+1)(2n+1)/6}} \right) \right].
\end{aligned}$$

6 Setting: median, two variables

Object: Here X and Y are **independent continuous symmetric** random variables for which the cdf F_X and F_Y satisfies $F_X(x) = F_Y(x + c)$.

Hypotheses:

- **Null Hypothesis** H_0: The median m_X for X is equal to the median m_Y for Y.
- **Alternative Hypothesis** H_1: The alternative hypothesis can take one of these three forms:
  (a) m_X is **strictly greater than** m_Y;
  (b) m_X is **strictly smaller than** m_Y;
  (c) m_X is **not equal to** m_Y.

Input: Random samples $x_1, \ldots, x_n$ for X and $y_1, \ldots, y_m$ for Y and significance level α. Note that n can be different from m.

Methodology:

- Compute the **Wilcoxon's rank sum statistic** W';

- Compute the critical region that depends on α; or

- Compute the p-value that depends on W'.

Output:

- Reject the null hypothesis if W' is contained in the critical region. Equivalently, reject the null hypothesis if the p-value is smaller than α.

- Do not reject the null hypothesis (i.e., test is inconclusive) otherwise.

7 Wilcoxon's rank sum statistic

Definition 3. Wilcoxon's rank sum statistic is

$$W' := \sum_{i=1}^{n} \text{Rank}(x_i),$$

where

- $\text{Rank}(x_i) \in \{1, \ldots, n+m\}$ is the rank of x_i among the $n+m$ joint samples of X and Y, with 1 being the smallest rank and $n+m$ being the largest rank. Ties receive a rank equal to the average of the ranks they span.

8 Theorem: median, two variables

Theorem 4. *(a) For the case $m_X > m_Y$,*

$$\begin{aligned} \textit{critical region} &= \left[\frac{n(n+m+1)}{2} + z_\alpha\sqrt{\frac{nm(n+m+1)}{12}}\ ,\ \infty\right), \\ \textit{p-value} &= 1 - \Phi\left(\frac{W' - \frac{n(n+m+1)}{2}}{\sqrt{\frac{nm(n+m+1)}{12}}}\right). \end{aligned}$$

(b) For the case $m_X < m_Y$,

$$\begin{aligned} \textit{critical region} &= \left(-\infty\ ,\ \frac{n(n+m+1)}{2} - z_\alpha\sqrt{\frac{nm(n+m+1)}{12}}\right], \\ \textit{p-value} &= \Phi\left(\frac{W' - \frac{n(n+m+1)}{2}}{\sqrt{\frac{nm(n+m+1)}{12}}}\right). \end{aligned}$$

(c) For the case $m_X \neq m_Y$, we have

$$\textit{critical region} \; = \left(-\infty \; , \; \frac{n(n+m+1)}{2} - z_{\alpha/2}\sqrt{\frac{nm(n+m+1)}{12}} \right] \cup \left[\frac{n(n+m+1)}{2} + z_{\alpha/2}\sqrt{\frac{nm(n+m+1)}{12}} \; , \; \infty \right),$$

$$\textit{p-value} \; = 2 \left[1 - \Phi \left(\frac{|W' - \frac{n(n+m+1)}{2}|}{\sqrt{\frac{nm(n+m+1)}{12}}} \right) \right].$$

9 Example: median, two variables

The weights of the contents of $n = 8$ and $m = 8$ tins of cinnamons packaged by Friendly Instructor Company and Evil Tyrant Company, respectively, is as follows:

x_i :	115.1	117.1	117.4	119.5	121.3	122.1	124.5	127.8
y_i :	117.2	122.1	123.5	125.3	125.6	126.5	127.9	129.8.

Can we reject the hypothesis that the weight of two (random) tins of cinnamons have the same median with significance level $\alpha = 0.05$?

10 Answer: median, two variables

Let's compute the joint rank of the samples:

x_i :	115.1	117.1	117.4	119.5	121.3	122.1	124.5	127.8
Ranks :	1	2	4	5	6	7.5	10	14
y_i :	117.2	122.1	123.5	125.3	125.6	126.5	127.9	129.8
Ranks :	3	7.5	9	11	12	13	15	16.

Therefore W' is equal to

$$W' = 1 + 2 + 4 + 5 + 6 + 7.5 + 10 + 14 = 49.5.$$

This is case (c), so we have

$$z_{\alpha/2}\sqrt{\frac{nm(n+m+1)}{12}} \;=\; (1.960)\sqrt{\frac{(8)(8)(8+8+1)}{12}} \;=\; 18.663.$$

So the critical region is

$$\left(-\infty\ ,\ 49.337\right] \cup \left[86.663\ ,\ \infty\right).$$

Since 49.5 is not contained in the critical region, the test is inconclusive.