Math 170S

Lecture Notes Section 8.4 [*][†]

Tests about proportions

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†]This notes is based on Hanbaek Lyu's and Liza Rebrova's notes from the previous quarter, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Setting: Bernoulli, one variable

Object: Y is a **Bernoulli** random variables with **unknown parameter** p.

Hypotheses:

- **Null Hypothesis** H_0: p is equal to p_0.
- **Alternative Hypothesis** H_1: The alternative hypothesis can take one of these three forms:
  (a) p is **strictly greater than** p_0;
  (b) p is **strictly smaller than** p_0;
  (c) p is **not equal to** p_0.

Input: Random samples $Y_1, \ldots, Y_n$ for Y (which are either 0 or 1) and significance level α.

Methodology:

- Compute the critical region that depends on α (and potentially $\overline{\mathrm{Y}}$); and

Output:

- Reject the null hypothesis if $\overline{\mathrm{Y}}$ is contained in the critical region.

- Do not reject the null hypothesis (i.e., test is inconclusive) otherwise.

2 Theorem: Bernoulli, one variable

Theorem 1. *(a) For the case $p > p_0$,*

$$\textit{critical region} \; = \left[p_0 + z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}} \;,\; \infty\right).$$

(b) For the case $p < p_0$,

$$\textit{critical region} \; = \left(-\infty \;,\; p_0 - z_\alpha \sqrt{\frac{p_0(1-p_0)}{n}}\right].$$

(c) For the case $p \neq p_0$,

$$\textit{critical region} \; = \left(-\infty \;,\; p_0 - z_{\alpha/2} \sqrt{\frac{p_0(1-p_0)}{n}}\right] \cup \left[p_0 + z_{\alpha/2} \sqrt{\frac{p_0(1-p_0)}{n}} \;,\; \infty\right).$$

3 Example: Bernoulli, one variable

Your instructor suspected that the dice used by a certain magical casino has been tampered with, so that the probability p of rolling a six with these dice is strictly higher than 1/6.

To validate his hypothesis, he played $n = 8000$ times, and he saw that six was rolled 1375 times.

Could he reject the hypothesis that the dice is fair with significance level $\alpha = 0.05$?

4 Answer: Bernoulli, one variable

Here we have

- **Null Hypothesis**: p is equal to $\frac{1}{6}$.
- **Alternative Hypothesis**: The alternative hypothesis is $p > \frac{1}{6}$.

So we have case (a), which gives us

$$
\begin{aligned}
z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}} &= z_{0.05}\sqrt{\frac{(1/6)(5/6)}{8000}} \\
&=(1.645)\sqrt{\frac{(1/6)(5/6)}{8000}} = 0.007.
\end{aligned}
$$

So the critical region is

$$\left[p_0 + z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}}\ ,\ \infty\right) \;=\; \left[\frac{1}{6} + 0.007\ ,\ \infty\right)$$
$$= \left[0.17367\ ,\ \infty\right)$$

Since the sample mean $\overline{\mathrm{Y}} = \frac{1375}{8000} = 0.171875$ is not contained in the critical region (albeit barely), the test is inconclusive.

Remark 2. Note that the textbook has an alternative formula for the critical region in Theorem 1, where

$$\sqrt{\frac{p_0(1-p_0)}{n}} \quad \text{is replaced by} \quad \sqrt{\frac{\overline{\mathrm{Y}}(1-\overline{\mathrm{Y}})}{n}}.$$

These two formulas yield approximately the same numerical result in practice.

The pros for lecture version is the critical region can be calculated *without* knowing the sample mean $\overline{\mathrm{Y}}$ in advance, and is consistent with Table 8.4-1 in the textbook.

The cons for lecture version is the formula is slightly inconsistent with the formula in Theorem 4 from Lecture Notes 7.3.

My conclusion is that pros outweighs cons, so we will use the lecture notes formula (unless indicated otherwise).

5 Setting: Bernoulli, two variables

Object: Y_1 and Y_2 are **independent Bernoulli** random variables with **unknown parameter** p_1 and p_2.

Hypotheses:

- **Null Hypothesis** H_0: p_1 is equal to p_2.
- **Alternative Hypothesis** H_1: The alternative hypothesis can take one of these three forms:
  (a) p_1 is **strictly greater than** p_2;
  (b) p_1 is **strictly smaller than** p_2;
  (c) p_1 is **not equal to** p_2.

Input: Significance level α, n_1 many random samples for Y_1, and n_2 many random samples for Y_2.

Methodology:

- Compute the critical region that depends on α and the given random samples.

Output:

- Reject the hypothesis if $\overline{\mathrm{Y}}_1 - \overline{\mathrm{Y}}_2$ is contained in the critical region.

- Do not reject the hypothesis (i.e., test is inconclusive) otherwise.

6 Theorem: Bernoulli, two variables

Theorem 3. *(a) For the case $p_1 > p_2$, the critical region is*

$$\left[z_\alpha\sqrt{\left(\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(1-\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}\ ,\ \infty\right).$$

(b) For the case $p_1 < p_2$, the critical region is

$$\left(-\infty\ ,\ -z_\alpha\sqrt{\left(\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(1-\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}\right].$$

(c) For the case $p_1 \neq p_2$, the critical region is

$$\left(-\infty\ ,-z_{\alpha/2}\sqrt{\left(\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(1-\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}\right] \cup$$
$$\left[z_{\alpha/2}\sqrt{\left(\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(1-\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}\ ,\ \infty\right).$$

7 Example: Bernoulli, two variables

There is a superstition among Dungeons and Dragons players that one needs to "pre-roll" the dice to get bad rolls out of the way.

The friendly instructor performed an experiment with one pre-rolled die and one new die.

- For the pre-rolled die, six was rolled 137 out of 800 observations.
- For the new die, that six was rolled 99 out of 600 observations.

Can he reject that the pre-rolling ritual is a mere superstition with significance level $\alpha = 0.05$?

8 Answer: Bernoulli, two variables

We are in case (c), so we have

$$\overline{\mathrm{Y}}_1 = \frac{137}{800}; \qquad \overline{\mathrm{Y}}_2 = \frac{99}{600};$$

$$z_{\alpha/2}\sqrt{\left(\frac{\overline{\mathrm{Y}}_1 + \overline{\mathrm{Y}}_2}{n_1 + n_2}\right)\left(1 - \frac{\overline{\mathrm{Y}}_1 + \overline{\mathrm{Y}}_2}{n_1 + n_2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$= (1.96)\sqrt{\left(\frac{\frac{137}{800} + \frac{99}{600}}{800 + 600}\right)\left(1 - \frac{\frac{137}{800} + \frac{99}{600}}{800 + 600}\right)\left(\frac{1}{800} + \frac{1}{600}\right)}$$

$$= 0.0016.$$

So the critical region is

$$\Big(-\infty\ ,\ -0.0016\Big] \ \cup\ \Big[0.0016\ ,\ \infty\Big).$$

Since $\overline{\mathrm{Y}}_1 - \overline{\mathrm{Y}}_2 = \frac{137}{800} - \frac{99}{600} = 0.00625$ is contained in the critical region, we reject the null hypothesis.

Remark 4. Note that the textbook has an alternative formula for the critical region in Theorem 3, where

$$\sqrt{\left(\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(1-\frac{\overline{\mathrm{Y}}_1+\overline{\mathrm{Y}}_2}{n_1+n_2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

is replaced by

$$\sqrt{\frac{\overline{\mathrm{Y}}_1(1-\overline{\mathrm{Y}}_1)}{n_1}+\frac{\overline{\mathrm{Y}}_2(1-\overline{\mathrm{Y}}_2)}{n_2}}.$$

These two formulas yield approximately the same numerical result in practice.

The lecture version is chosen to be consistent with Table 8.4-2.

This choice is slightly inconsistent with the formula in Theorem 6 from Lecture Notes 7.3.

My conclusion is that being consistent with the textbook will reduce logistical problems in the long run; hence the decision.