Math 170S

Lecture Notes Section 8.1 [*,†]

Tests about one mean

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NOTE: Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

[*]Version date: Monday 16th November, 2020, 11:07.

[†]This notes is based on Hanbaek Lyu's and Liza Rebrova's notes from the previous quarter, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Example: Simple alternative hypothesis

The mayor of Duckburg wants to check if COVID-19 has spread in the city.

She plans to do so by checking the average amount of coronavirus antibody from $n = 16$ random citizens. Her expert team told her that

- If Duckburg is coronavirus-free, then the amount of coronavirus antibody is a normal random variable $N(50, 36)$ with mean 50 and variance 36.

- If the coronavirus has spread in Duckburg, then the amount of coronavirus antibody is a normal random variable $N(55, 36)$ with mean 55 and variance 36.

Here strategy is thus

- If the average amount of antibody is higher than 53, she will **reject** the hypothesis that Duckburg is coronavirus-free, and calls for a lockdown.
- If the average amount of antibody is lower than 53, she will **not reject** the hypothesis that Duckburg is coronavirus-free, and opens the beaches.

What is the probability that she makes the wrong decision?
(i.e., calling for lockdown when the city is coronavirus-free, or opening beaches when coronavirus has spread.)

2 Settings: Simple Alternative Hypothesis (SAH)

- **Object:** X is an **unknown** random variables with **unknown mean** μ but **known variance** σ^2.

- **Hypotheses:** You know that μ is equal to one of these two given values:

  - **Null Hypothesis** H_0: μ is equal to μ_0. This hypothesis is generally assumed to be true until evidence indicates otherwise.

  - **Alternative Hypothesis** H_1: μ is equal to μ_1. This hypothesis generally states something is happening, that a new theory is true instead of an old one.

- **Input:** Random samples $X_1, \ldots, X_n$ for X.
- **Methodology:**
  - Compute the sample mean $\overline{\mathrm{X}}$. This is called the **test statistics.**
  - A subset of real numbers called the **critical region**.
- **Output:**
  - **Reject the null hypothesis** if $\overline{\mathrm{X}}$ is contained in the critical region;
  - **Do not reject the null hypothesis** if $\overline{\mathrm{X}}$ is not contained in the critical region. This means **the test is inconclusive**.

3 Answer for Example: SAH

For this example,

- **Hypotheses:**
  - **Null Hypothesis**: μ is equal to 50 (there is no coronavirus).
  - **Alternative Hypothesis**: μ is equal to 55 (coronavirus is spreading).
- **Methodology:** The **critical region** is

$$[53, \infty),$$

since we reject the null hypothesis if $\overline{\mathrm{X}}$ is greater than 53, and do not reject if $\overline{\mathrm{X}}$ is less than 53.

Let α be the probability that the null hypothesis H_0 is true, but H_0 is rejected (lockdown is called when there is no coronavirus).

Since the null hypothesis is true,

$$\overline{\mathrm{X}} \text{ has mean } \mu_0 = 50 \text{ and variance } \tfrac{\sigma^2}{n} = \tfrac{36}{16}.$$

So α is equal to

$$\alpha = P[\overline{\mathrm{X}} \geq 53] \; = 1 - \Phi\left(\frac{53 - \mu_0}{\sigma/\sqrt{n}}\right) \; = \; 1 - \Phi\left(\frac{53 - 50}{\frac{6}{4}}\right)$$

$$= 1 - \Phi(2) \; = \; 0.0228 \, .$$

(Here Φ is the cdf of standard normal random variable.)

Let β be the probability that the null hypothesis is false, but the null hypothesis is not rejected (the beach is open when the coronavirus is spreading).

Since the alternative hypothesis is true,

$$\overline{\mathrm{X}} \text{ has mean } \mu_1 = 55 \text{ and variance } \tfrac{\sigma^2}{n} = \tfrac{36}{16}.$$

So β is equal to

$$\beta = P[\overline{\mathrm{X}} < 53] \; = \Phi\left(\frac{53-\mu_1}{\sigma/\sqrt{n}}\right) \; = \; \Phi\left(\frac{53-55}{\frac{6}{4}}\right) \; = \; \Phi(-4/3)$$

$$= 0.0913.$$

The probability of making wrong decision is thus

$$\alpha + \beta \; = \; 0.0228 + 0.0913 \; = \; 0.1141\,.$$

4 Type I and Type II error

- **Type I error** is when we reject H_0 eventhough H_0 is true. It is also called **false positive**.

  The probability of type I error is denoted by α, and is often called the **significance level** of the test.

- **Type II error** is when we do not reject H_0 eventhough H_0 is false. It is also called **false negative**.

  The probability of type II error is denoted by β.

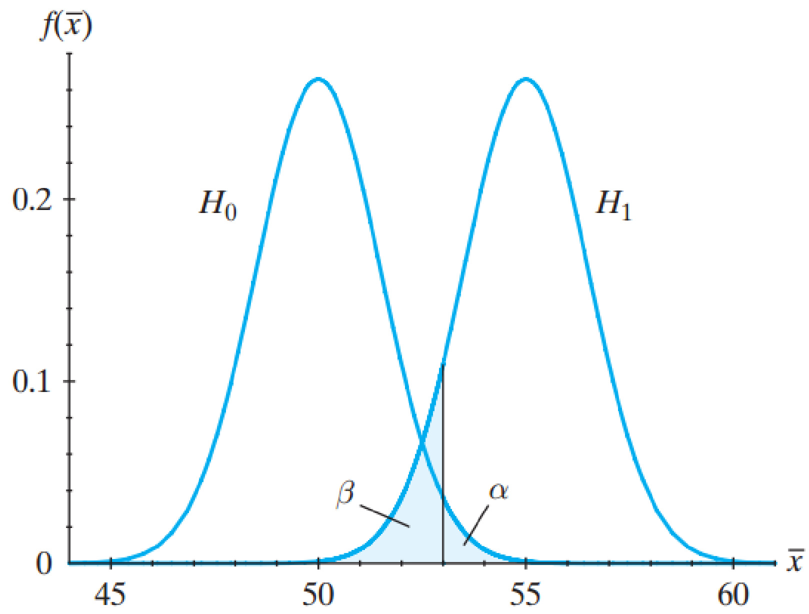

Figure 1: The error probabilities α and β for Example Duckburg.

5 Example: Composite Alternative Hypothesis (CAH)

The mayor of Duckburg now learns that the amount of coronavirus antibody is a normal random variable with unknown mean μ and variance 100. The hypotheses are

- If there is no coronavirus in Duckburg, then μ is equal to 60 (**null hypothesis**).
- If the coronavirus has spread in Duckburg, then μ is strictly greater than 60 (**alternative hypothesis**).

From experiments she found that the average amount of coronavirus antibody from $n = 52$ random citizens is 62.75. Compute the p-value of this experiment.

6 p-value

p-value is the probability of obtaining results at least as extreme as the sample mean, given that H_0 is correct.

Note that

The smaller the p value is, the more likely that H_0 is false.

Thus we **reject the null hypothesis** only when the p-value is **smaller than** a constant α (the **significance level**) that we decided beforehand.

It is common practise for scientific papers to set $\alpha = 0.05$.

7 Continued example: CAH

If H_0 is true (Duckburg is coronavirus-free), then

$\overline{\mathrm{X}}$ is a normal random variable with mean $\mu_0 = 60$ and variance $\frac{100}{52}$.

The p-value is thus

$$\begin{aligned} p - \text{value} &= P[\overline{\mathrm{X}} \geq 62.75 \mid H_0 \text{ is true}] \\ &= 1 - \Phi\left(\frac{62.75 - \mu_0}{\sigma/\sqrt{n}}\right) \\ &= 1 - \Phi\left(\frac{62.75 - 60}{10/\sqrt{52}}\right) \\ &= 0.0237. \end{aligned}$$

Suppose that the significance level α is 0.05. In this case, p-value (0.0237) is less than α, and thus we **reject the null hypothesis**. (This means that coronavirus is already spreading.)

8 Settings: Composite altenative hypothesis

- **Object:** X is an **unknown** random variable with **unknown mean** μ but **known variance** σ^2.
- **Hypotheses:**
  - **Null Hypothesis** H_0: μ is equal to μ_0.
  - **Alternative Hypothesis** H_1: The alternative hypothesis can take one of these three forms:
    (a) μ is **strictly greater than** μ_0;
    (b) μ is **strictly smaller than** μ_0;
    (c) μ is **not equal to** μ_0.
- **Input:** Random samples $X_1, \ldots, X_n$ for X and **significance level** α.

- **Methodology:**
  - Compute the critical region that depends on α; or
  - Compute the p-value that depends on $\overline{\mathrm{X}}$.
- **Output:**
  - **Reject the null hypothesis** if $\overline{\mathrm{X}}$ is contained in the critical region.

    Equivalently, **reject the null hypothesis** if the p-value is smaller than α.
  - **Do not reject the null hypothesis** otherwise (test is inconclusive).

Remark 1. The critical region is given by the values $\overline{\mathrm{X}}$ such that the p-value is less than α.

9 Theorem: CAH

Theorem 2. *(a) For alternative hypothesis: μ is* ***strictly greater*** *than μ_0,*

$$\begin{aligned} \textit{critical region} &= \left[\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}\ ,\ \infty\right), \\ \textit{p-value} &= 1 - \Phi\left(\frac{\overline{\mathrm{X}} - \mu_0}{\sigma/\sqrt{n}}\right). \end{aligned}$$

(b) For alternative hypothesis: μ is ***strictly smaller*** *than μ_0,*

$$\begin{aligned} \textit{critical region} &= \left(-\infty\ ,\ \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}\right], \\ \textit{p-value} &= \Phi\left(\frac{\overline{\mathrm{X}} - \mu_0}{\sigma/\sqrt{n}}\right). \end{aligned}$$

(c) For alternative hypothesis: μ is ***not equal*** *to μ_0,*

$$critical\ region = \left(-\infty\ ,\ \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right] \cup \left[\mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ \infty \right),$$

$$p\text{-}value = 2\left[1 - \Phi\left(\frac{|\,\overline{X} - \mu_0|}{\sigma/\sqrt{n}} \right) \right].$$

Remark 3. The theorem is derived using the central limit theorem. Check the textbook for a proof.

Remark 4. It is easy to mistake the critical region and the confidence interval (Section 7.1),

$$\text{critical region} \;=\; \Big(-\infty\;,\;\mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\Big] \;\cup\; \Big[\mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\;,\;\infty\Big),$$

$$\text{confidence interval} \;=\; \left[\overline{\mathrm{X}} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\;,\;\overline{\mathrm{X}} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$

However, there are two major differences:

- The critical region is defined using μ_0, while confidence interval is defined using the sample mean $\overline{\mathrm{X}}$;
- The critical region is the regime where the events are unlikely to be observed, while the confidence interval is the regime where the events are likely to be observed.

10 Another example: CAH

The power level of a random ninja is a random variable with unknown mean μ and with variance 100.

- The null hypothesis H_0 is μ is equal to 100.
- The alternative hypothesis is that μ has mean strictly lower than 100.

The sample mean $\overline{\mathrm{X}}$ of the power level of $n = 16$ random ninjas is $\overline{\mathrm{X}} = 90$. Do we accept or reject H_0 at the 5% significance level?

11 Answer to another example: CAH

This is case (b). Then

$$z_\alpha \frac{\sigma}{\sqrt{n}} \;=\; z_{0.05}\frac{10}{\sqrt{16}} = (1.645)(\frac{5}{2}) = 4.1125$$

So the critical region is

$$\left(-\infty\,,\;\mu_0 - z_\alpha\frac{\sigma}{\sqrt{n}}\right] \;=\; \left(-\infty\,,\;100-4.1125\right] \;=\; (-\infty, 95.8875].$$

Since the sample mean $\overline{\mathrm{X}}$ is contained in the critical region, we choose to reject the null hypothesis.

Let's try to solve this question by using p-value instead.

The p-value is equal to

$$\Phi\left(\frac{\overline{\mathrm{X}} - \mu_0}{\sigma/\sqrt{n}}\right) = \Phi\left(\frac{90 - 100}{10/\sqrt{16}}\right) = \Phi(-4) \approx 0.$$

This p-value is clearly smaller than $\alpha = 0.05$, so we again choose to reject the null hypothesis.

12 Compositive alternative hypothesis: unknown variances

Only **Object** changes, **Hypotheses**, **Input**, **Methodology**, **Output** stay the same.

- **Object:** X is an **unknown** random variables with **unknown mean** μ and **unknown variance**.
- **Hypotheses:**
  - **Null Hypothesis** H_0: μ is equal to μ_0.
  - **Alternative Hypothesis** H_1 is one of these three:
    (a) μ is **strictly greater than** μ_0;
    (b) μ is **strictly smaller than** μ_0;

(c) μ is **not equal to** μ_0.

- **Input:** Random samples $X_1, \ldots, X_n$ for X and significance level α.

- **Methodology:**

  - Compute the critical region that depends on α; and

  - Compute the p-value that depends on $\overline{\mathrm{X}}$.

- **Output:**

  - Reject the null hypothesis if $\overline{\mathrm{X}}$ is contained in the critical region. Equivalently, reject the null hypothesis if the p-value is smaller than α.

  - Do not reject the null hypothesis (i.e., test is inconclusive) otherwise.

13 Theorem: CAH unknown variances

Theorem 5. *If $n > 30$, apply the formula in Theorem 2 with sample variance s^2 substituting the real variance σ^2.*

If $n \leq 30$ but X is ***normal*** *RV, apply the following formula:*

(a) *For alternative hypothesis: μ is* ***strictly greater*** *than μ_0,*

$$\text{critical region } = \left[\mu_0 + t_\alpha(n-1)\frac{s}{\sqrt{n}}\ ,\ \infty\right),$$

$$p\text{-value } = P\left[T\ \geq\ \frac{\overline{\mathrm{X}} - \mu_0}{s/\sqrt{n}}\right].$$

(b) *For alternative hypothesis: μ is* ***strictly smaller*** *than μ_0,*

$$\textit{critical region} \;=\; \left(-\infty\ ,\ \mu_0 - t_\alpha(n-1)\frac{s}{\sqrt{n}}\right],$$

$$\textit{p-value} \;=\; P\left[T \;\leq\; \frac{\overline{\mathrm{X}}-\mu_0}{s/\sqrt{n}}\right].$$

(c) *For alternative hypothesis: μ is* ***not equal*** *to μ_0,*

$$\textit{critical region} \;=\; \left(-\infty\ ,\ \mu_0 - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right] \cup \left[\mu_0 + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\ ,\ \infty\right),$$

$$\textit{p-value} \;=\; 2\,P\left[T \;\geq\; \frac{|\,\overline{\mathrm{X}}-\mu_0|}{s/\sqrt{n}}\right].$$

Here T is the t distribution with $n-1$ degrees of freedom.

14 Example: CAH unknown variances

Suppose that the power level X of a Pokemon is a **normal** random variable with **unknown mean** μ and **unknown variance**.

- Null hypothesis is that μ is equal to 4.
- Alternative hypothesis is that μ is not equal to 4.

Your friendly instructor caught nine Pokemons. Their power level has sample mean 4.3 and sample standard deviation 1.2.

Do we reject the null hypothesis at the 10% significance level?

15 Answer: CAH unknown variances

This is case (c), so we have

$$t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}} \;=\; t_{0.05}(8)\frac{1.2}{\sqrt{9}} = (1.860)(\frac{1.2}{3}) = 0.744.$$

So the critical region is

$$\left(-\infty\,,\;\mu_0 - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right] \cup \left[\mu_0 + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\,,\;\infty\right),$$

which is equal to

$$\left(-\infty\,,\;3.256\right] \cup \left[4.744\,,\;\infty\right),$$

Since the sample mean $\overline{\mathrm{X}}$ is not contained in the critical region, this test is inconclusive.

Let's try to solve this question by using p-value instead.

The p-value is equal to

$$2\,P\left[T\ \geq\ \frac{|\,\overline{\mathrm{X}}-\mu_0|}{s/\sqrt{n}}\right]\ =\ 2\,P\left[T\ \geq\ \frac{|4.3-4.0|}{1.2/\sqrt{9}}\right]$$

$$=\ 2P[T \geq 0.75]\ \approx\ 0.50.$$

This p-value is larger than $\alpha = 0.1$, so again the test is inconclusive.