Math 170S

Lecture Notes Section 6.0 [*†]

Probability theory: recap

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NOTE: The notes is a summary for materials discussed in the class and is not supposed to substitute the textbook. Materials that appear in the textbook but do not appear in the lecture notes might still be tested. Please send me an email if you find typos.

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[†] This notes is based on Hanbaek Lyu's and Liza Rebrova's notes from the past quarters, and I would like to thank them for their generosity. "*Nanos gigantum humeris insidentes* (I am but a dwarf standing on the shoulders of giants)".

1 Probability

Why should we study probability?

> "*There is nothing certain; but the uncertain.*"

That is why, in order to not let *uncertainty* being in control of our life, we need to understand the true nature of *uncertainty* and use this knowledge to achieve our life's objectives.

2 Probability: example

When you throw a six-sided dice, there are six possible outcomes:

- With probability $\frac{1}{6}$, we have number 1 as the outcome;
- With probability $\frac{1}{6}$, we have number 2 as the outcome;
- $\ldots$;
- With probability $\frac{1}{6}$, we have number 6 as the outcome.

These are all the six outcomes, and the sum of the probabilities is equal to 1.

3 Probability space

The **outcome space** is the collection of all possible outcomes of a random experiment, usually denoted by S.

In the dice example, the random experiment is a dice-throw, and the outcome space is $\{1, 2, 3, 4, 5, 6\}$.

An **event** is a subset of outcomes. An event A **has occurred** if the outcome of the random experiment **is contained** in A.

Here are some possible events for the dice example:

- $A := \{1, 2, 4, 5\}$;
- $B := \{\text{the outcome is even}\} = \{2, 4, 6\}$.
- $C := S = \{1, 2, 3, 4, 5, 6\}$.
- $D := \{\} = \varnothing$ (empty set).

4 Probability function

The **probability function (measure)** P assigns to each event A a real number $P(A)$. The probability function must satisfy these three rules:

- $P[A] \geq 0$ for any event A;
- $P[S] = 1$;
- Let $A_1, A_2, \ldots A_k$ be k events such that $A_i \cap A_j$ is empty.

  $$P[A_1 \cup A_2 \cup \ldots \cup A_k] = P[A_1] + P[A_2] + \ldots + P[A_k].$$

The probability measure P in the dice example is

$$P[\{1\}] = P[\{2\}] = P[\{3\}] = P[\{4\}] = P[\{5\}] = P[\{6\}] = \frac{1}{6},$$

with probabiltiy for other events could be derived from the three rules above.

5 Uniform probability measure

Suppose that the outcome space S has m distinct elements $x_1, \ldots, x_m$. The **uniform probability measure** is

$$P[x_1] = P[x_2] = \ldots = P[x_m] = \frac{1}{m},$$

so all outcomes are equally likely.

The probability measure in the dice example is a uniform probability measure with six outcomes.

6 Random variable

A **random variable** X is just a random number.

In the dice example, let X be the outcome of the dice-throw.

- X is equal to 1 with probability $\frac{1}{6}$;
- X is equal to 2 with probability $\frac{1}{6}$;
- X is equal to 3 with probability $\frac{1}{6}$;
- X is equal to 4 with probability $\frac{1}{6}$;
- X is equal to 5 with probability $\frac{1}{6}$;
- X is equal to 6 with probability $\frac{1}{6}$.

7 Mean

Let X be a random variable with distinct outcomes $x_1, x_2, \ldots, x_n$. The **mean (expected value)** of X is

$$E[X] \; := x_1 P[X = x_1] + x_2 P[X = x_2] + \ldots + x_n P[X = x_n].$$

In the dice example, the mean of X is

$$E[X] = 1 \times \frac{1}{6} \; + \; 2 \times \frac{1}{6} \; + \; \ldots \; + \; 6 \times \frac{1}{6} = 3.5.$$

One should think of the mean $E[X]$ as how much you would earn *on average* from gambling in a casino, *assuming you play the same game over and over again*[1]. Here X is the (random) earning you make from one round of the game.

[1]If this somehow happens to you in real life, it might mean that you have a gambling addiction; please seek help immediately.

8 Linearity of mean

Lemma 1. *For any real numbers a, b,*

$$E[aX + b] = aE[X] + b.$$

This property will be used frequently throughout this quarter.

9 Variance

The **variance** of X is

$$\mathrm{Var}[X] := E[(X - E[X])^2] = E[X^2] - (E[X])^2.$$

In the dice example, the variance of X is equal to

$$\begin{aligned}\mathrm{Var}[X] =&(1-3.5)^2 \times \frac{1}{6} \ + \ (2-3.5)^2 \times \frac{1}{6} \ + \ (3-3.5)^2 \times \frac{1}{6} + \\ &(4-3.5)^2 \times \frac{1}{6} \ + \ (5-3.5)^2 \times \frac{1}{6} \ + \ (6-3.5)^2 \times \frac{1}{6} \\ =&\frac{17.5}{6} = \frac{105}{36}.\end{aligned}$$

The variance of X quantifies the *risk* that you take when you gamble in a casino and play the same game over and over again. The larger the variance, the larger the difference between your real (random) earning from the expected earning $E[X]$.

10 Bernoulli random variable

Let p be a real number satisfying $0 \leq p \leq 1$. The **Bernoulli random variable** X with parameter p is

$$X := \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1 - p. \end{cases}$$

11 Normal random variable

The **standard normal** random variable X is given by

$$P[X \leq x] \quad = \quad \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

It has mean 0 and variance 1.

The **normal** random variable X with μ and σ^2 is

$$P[X \leq x] \quad = \quad \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right) dz.$$

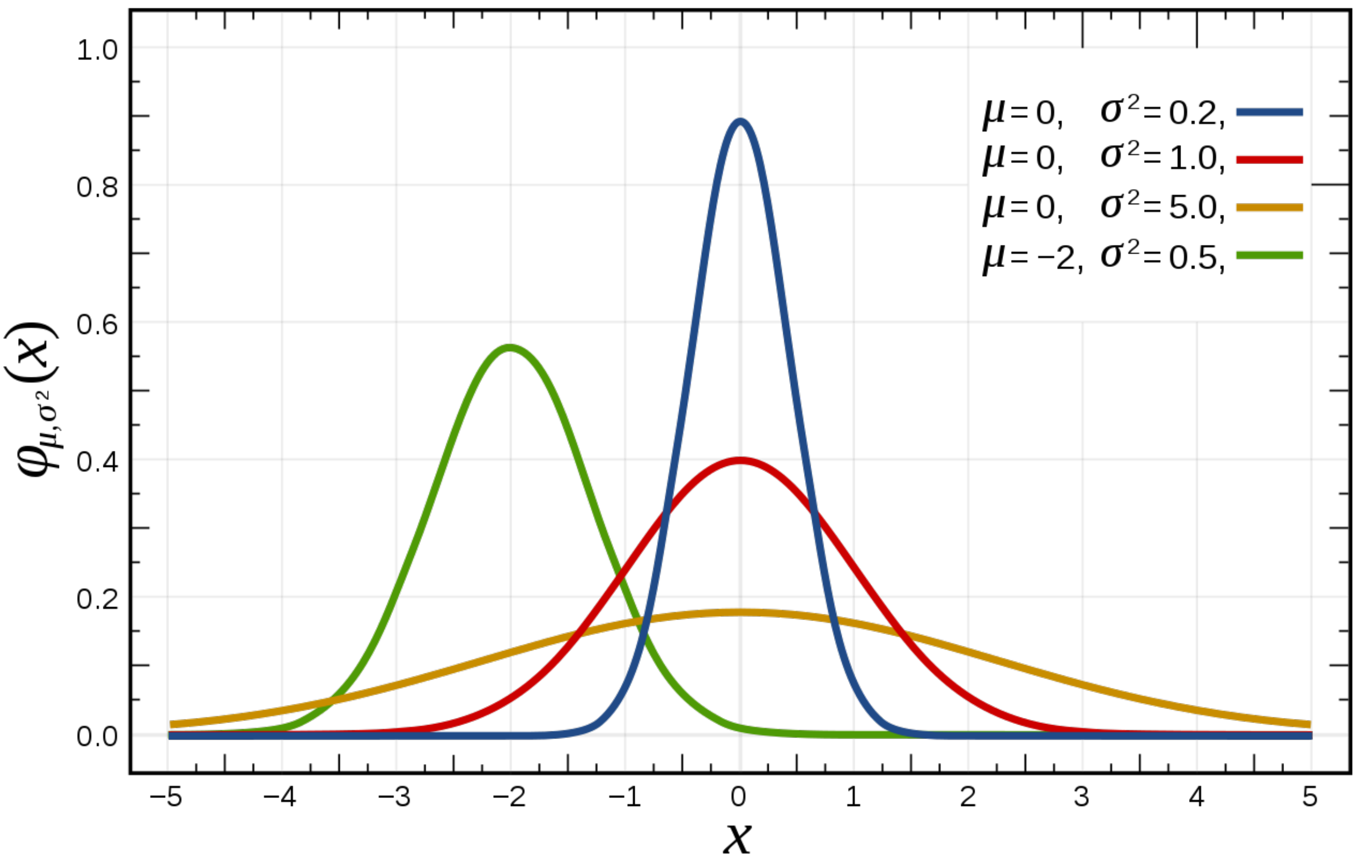

Figure 1: The sketch of the probability density function of various normal distributions, taken from Wikipedia. Note the shape of the curves, which resembles a bell.