Chapter 9

Binary relations and relationships [1]

This chapter introduces the object type *binary relation.* Before defining this object, we first introduce the related concept of a *binary relationship.*

9.1 Binary relationships

In everyday usage, we talk about relationships between two people or two objects such as “Bob and Jane are classmates”, “Antoine is the father of Katelyn”, or “The key is inside the mailbox”. Here each of the phrases “are classmates” or “is the father of” or “is inside of” specifies a relationship that holds between the individuals or objects mentioned in the sentence.

In the mathematical universe, a relationship that may hold between two mathematical objects is called a *binary relationship.*

Remark 9.1. The word *binary* is an adjective that means “pertaining to the number two”. A binary relationship involves *two objects.* A ternary relationship involves three objects, and a k-ary relationship involves k objects. We will be focusing on binary relationships in this chapter.

You may be familiar with the word *binary* as an alternate system for writing numbers, using just two symbols (usually 0 and 1) to represent all nonnegative integers. Binary relationships have nothing to do with binary notation for numbers, except that both involve the number two.

Example 9.2. Some binary relationships for a pair of real numbers *real numbers*:

[1] Version 1-23-2023

a. **x is less than y**. This is abbreviated $x < y$.

b. **x is less than or equal to y**. This is abbreviated $x \leq y$.

c. **y is the square of x.** This is abbreviated by $y = x^2$.

d. **x is within 1 of y**, that is, $|x - y| \leq 1$.

e. **y is the image of x under f**, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is some fixed function. This relationship is abbreviated $y = f(x)$. Note that the relationship "y is the square of x" is a special case of this one.

f. **x and y satisfy a specified equation or inequality.** Any equation or inequality involving x and y, "$x^2 + y^3 = 1$" or "$(x + y)^2 \leq x^3$" or $|x - y| \leq 1$, as above, defines a relationship between x and y.

Example 9.3. Some binary relationships between pairs m, n of *integers*:

a. **m is the successor of n.** This means that $m = n + 1$.

b. **m is a divisor of n.** This is abbreviated $m|n$, and means that n/m is an integer.

c. **m is coprime to n.** This means that m and n have no common divisor greater than 1.

d. m and n are **congruent modulo k** abbreviated as $m \equiv_k n$ or $m \equiv n \mod k$. Here k is a fixed positive integer, and the meaning is that $m - n$ is a multiple of k.

Example 9.4. Some binary relationships betwwen pairs A, B of *sets*.

a. **A is a subset of B.** This is abbreviated $A \subseteq B$.

b. **A and B are intersecting**. This means $A \cap B \neq \emptyset$.

c. **A is disjoint from B.** This means $A \cap B = \emptyset$.

d. **A and B have the same size.** This is abbrevated $|A| = |B|$.

Example 9.5. Some binary relationships between pairs k, ℓ of lists.

a. **k is a shuffle of ℓ.** This means that every object that appears in either k or ℓ, appears as an entry of both lists the same number of times. For example $(1, 1, 1, 2, 2, 3)$ is a shuffle of $1, 2, 3, 1, 2, 1)$.

b. k **and** ℓ **have the same set of entries.** We can form a set from any list consisting of the objects that appear as entries in the list. For example $(1, 1, 4, 2, 1, 2)$ has entry set $\{1, 2, 4\}$. Thus $(1, 1, 4, 2, 1, 2)$ and $(4, 4, 2, 2, 1, 1)$ have the same set of entries.

c. k **is a prefix of** ℓ**.** This means that there is a list k' such that $\ell = k * k'$, where $*$ is the concatentation operation.

In all of the above relationships, the two objects being related are of the same type. We refer to such a relationship as a **homogeneous relationship**. There are also relationships that may hold between objects of different types.

Example 9.6. Some relationships between different types of objects.

a. a **is an entry of** ℓ**.** For a list ℓ of integers and an integer a, a **is an entry of** ℓ means that a appears somewhere in the list ℓ.

b. b **is a root of** f**.** For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $b \in \mathbb{R}$ we say that b **is a root of** f provided that $f(b) = 0$.

c. z **is a lower bound for** A**.** If A is a subset of $\mathbb{R}$ and $z \in \mathbb{R}$ we say that z is a **lower bound** for A provided that for all $x \in A$, $z \leq x$.

Formally, a binary relationship on the pair of sets S,T is specified by a predicate $A(x, y)$ involving a pair of objects $(x, y) \in S \times T$. We say this is **binary relationship between S and T** or **from S to T**.

In the case $S = T$, the relationship $A(x, y)$ is **homogeneous** and we also say that $A(x, y)$ is a *binary relationship on S*.

For example the relationship "coprime to" introduced above is a homogeneous binary relationship on $\mathbb{Z}$, and the "is a root of" relationship is a relationship between $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}$.

9.2 Binary relations

Having defined binary relationships we now define the related mathematical object *Binary relation.*

Definition 9.7. A **binary relation** R is a 3-tuple (S, T, P) where:

- S is a set called the **source** of R, denoted **Source**(R)
- T is a set called the **target** of R, denoted **Target**(R)

- P is a subset of **Source**$(R) \times$ **Target**(R) called the **pair-set** of R, denoted **pairs**(R).

If R is a binary relation with source S and target T we say that R is:

- a **binary relation from S to T**, or
- a **binary relation between S and T**, or
- a **binary relation on $S \times T$**.

If the source and target are the same set S so that the relationship is **homogenous** we say that **R is a binary relation on S**. In this case, we sometimes refer to S as the **ground set** of R, denoted **Ground**(R), and usually denote R as an ordered pair (S, P) where S is the ground set and $P \subseteq S \times S$, rather than as the triple (S, S, P).

Example 9.8. Suppose S is the set $\{1, 2, 3, 4, 5, 6\}$ and let T be the set consisting of intervals $\{[1, 1], [1, 3], [2, 4], [1, 4], [3, 4], [6, 6]\}$. Consider the relationship between $s \in S$ and $I \in T$, defined by the condition $s \in I$. The associated relation R on $S \times \mathcal{P}(S)$ has

$$\begin{aligned}\textbf{pairs}(R) \quad = \{ \quad & (1, [1, 1]), (1, [1, 3]), (1, [1, 4]), (2, [1, 3]) \\ & (2, [2, 4]), (2, [1, 4]), (3, [1, 3]), (3, [2, 4]), (3, [3, 4]), \\ & (3, [1, 4]), (4, [1, 4]), (4, [2, 4]), (4, [3, 4]), (6, [6, 6])\}.\end{aligned}$$

Correspondence between binary relations and binary relationships. Every binary relationship is paired with a unique binary relationship.

- If $A(x, y)$ is a predicate defining a binary relationship between sets and S and T, then the associated relationship is the relation R_A between S and T where **pairs**$(R_A) = \{(x, y) \in S \times T : A(x, y)\}$. In words, **pairs**$(R_A)$ is the set of solutions for the predicate $A(x, y)$.

- If R is a binary relation between S and T then the predicate "$(x, y) \in$ **pairs**(R)" defines the associated binary relationship.

Every example of a relationship in Section 9.1 corresponds to a relation.

Specifying a relation To specify a relation R we must specify the sets S and T, and also the set $\mathbf{pairs}(R)$. Since S,T and $\mathbf{pairs}(R)$ are all sets, we can specify each of them using our notation for sets.

Example 9.9. 1. Let R be the relation between $S = \{1, 2, 3, 4, 5\}$ and $T = \{6, 7, 8, 9, 10\}$ with pair set $\{(1, 6), (1, 7), (1, 8), (2, 7), (3, 8), (3, 10), (5, 6), (5, 9), (5, 10)\}$. In this example each of the three sets is specified using the list representation of the sets.

2. Let Q be the relation on the set $\mathbb{N}$ with $\mathbf{pairs}(Q) = \{(m, n) : m|n\}$. Thus $(4, 12), (6, 30) \in Q$ but $(7, 34) \notin Q$.

9.2.1 Pictorial representations of a relation

As with any mathematical object, mathematicians seek useful ways to visualize a relation. Here are some possible ways to represent a relation pictorially.

- If the relation is between two small finite sets: We can use a point and arrow diagram with the elements of the source arranged on the left and those of the target on the right (similar to diagrams used for functions.) See Figure 9.1. If the sets are large or infinite, we can only draw a small portion of the diagram.

- If the relation is a homogeneous relation on a small finite set we can draw a diagram as above with the set repeated on the left or right. Alternatively, we can draw a diagram with only one copy of the set. See Figure 9.2.

- Even if the sets are not finite, we can use a picture as above, but draw only a part of the full picture.

- If the relation is between real numbers, then we can use the usual $x - y$ plane drawing to represent a relation. See Figure 9.2.1.

We introduce some basic notation relating to relations:

Definition 9.10. Let $R = (S, T, \mathbf{pairs}(R))$ be a relation.

- A member $(s, t) \in \mathbf{pairs}(R)$ is called a **pair of** R.

- sRt means $(s, t) \in R$.

- $s\not{R}t$ means $(s, t) \notin R$.

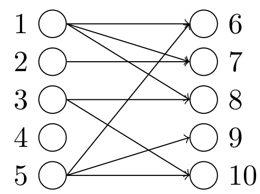

Figure 9.1: Diagram representing Example 9.9(1)

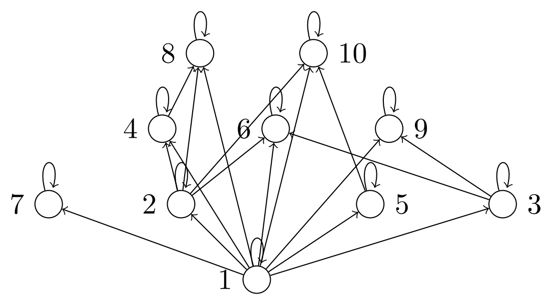

Figure 9.2: Diagram representing the portion of Example 9.9(2) for the integers $\{1, \ldots, 10\}$

- For $s \in S$, $R(s)$ is the set $\{t \in T : sRt\}$.
- For $X \subseteq S$, $\mathbf{R(X)}$ is equal to $\bigcup_{s \in S} R(s)$.

Sometimes we introduce a special symbol such as $\sim$ and write $x \sim y$ instead of xRy. Other symbols commonly introduced to denote a particular relationship are $\equiv$ and $\preceq$.

Remark 9.11. A common error in use of terminology is to a member $(s, t) \in$ $\mathbf{pairs}(R)$ as "a relation". The relation is the triple $R = (S, T, \mathbf{pairs}(R))$ and (s, t) is a *pair of* R.

Example 9.12. To illustrate the above notation, for the relation in Example 9.8 we have $3R[3, 4]$ and $1 \not R [2, 4]$. Also $R(5) = \emptyset$, $R(3) = \{[1, 3], [2.4], [1, 4], [3, 4]\}$ and $R(\{1, 2, 6\}) = \{[1, 1], [1, 3], [1, 4], [2, 2], [2, 4], [6, 6]\}$.

Exercise 9.2.1. For each of the example relationships in 9.1, consider the associated relation R and give an example of (a) a pair $(x, y) \in \mathbf{Source}(R) \times$ $\mathbf{Target}(R)$ such that xRy, and (b) a pair $(x, y) \in \mathbf{Source}(R) \times \mathbf{Target}(R)$ such that $x \not R y$.

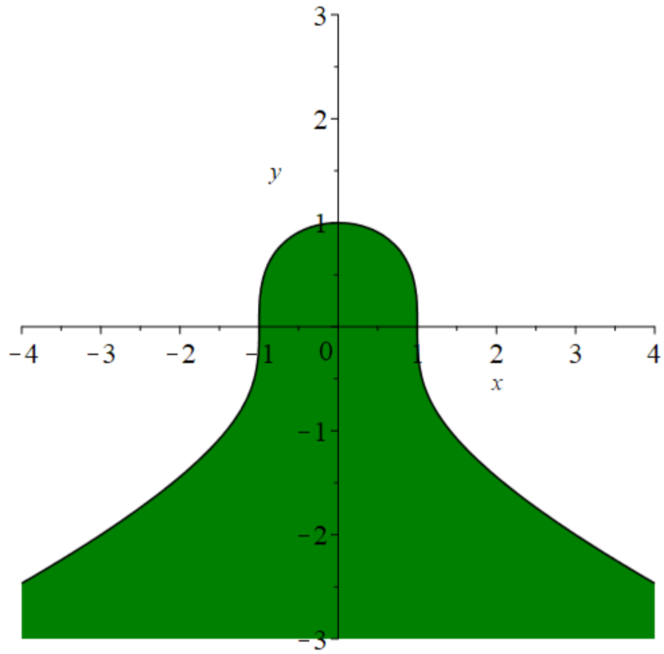

Figure 9.3: Relation $\{(x, y) \in \mathbb{R}^2 : x^2 + y^3 \leq 1\}$ depicted in the $x - y$ plane.

Remark 9.13. Mathematicians usually don't distinguish between a "binary relation" and the corresponding "binary relationship", and simply use the word "binary relation" for both. In most books, when the author refers to the "$<$ relation" for real numbers, he or she might mean the "$<$" *relationship* or the *relation* $\{(x, y) \in \mathbb{R}^2 : x < y\}$. It is up to the reader to use the context to figure out which is meant.

In this introductory book, we distinguish between a "relationship" and "relation" because they represent two different (though closely connected) concepts.

Here are a few additional important relations.

Definition 9.14. For sets S and T, the **complete relation** $K(S, T)$ is the relation $S \times T$ whose pair set is all of $S \times T$. When $S = T$ we abbreviate $K(S, S)$ by the notation $K(S)$.

Example 9.15. If $f : S \longrightarrow T$ is a function, then we can define the relationship $y = f(x)$ with source S and target T. The associated relation is named R_f, and has source S, target T and $\mathbf{pairs}(R_f) = \{(x, y) \in S \times T : y = f(x)\}$.

Advanced Remark 9.16. As we have defined them, the function f and the relation R_f are different, but closely related, objects. In many mathematics books, the function f and relation R_f are viewed as being the same object. Such books define *relation* before defining *function*, and define a function to be a relation with the property that all of the pairs have different first coordinates.

In this way we can think of a function as a special type of relation.

Remark 9.17. While $\mathbf{Source}(R)$ and $\mathbf{Target}(R)$ are a necessary part of the definition of R, the most important part of R is the set $\mathbf{pairs}(R)$. Mathematicians often abbreviate the set $\mathbf{pairs}(R)$ by R. This is an abuse of notation since the proper meaning of R is the list $(\mathbf{Source}(R), \mathbf{Target}(R), \mathbf{pairs}(R))$. When first learning about relations, it is better to use the notation properly and use $\mathbf{pairs}(R)$ for the set of ordered pairs.

Advanced Remark 9.18. There are other kinds of relations other than binary relations. For any positive integer k, a **k-ary relation** consists of a sequence $S_1, S_2, \ldots, S_k$ of sets and a set of lists $(s_1, \ldots, s_k)$ where for each i, $s_i \in S_i$. Binary relations are the same as 2-ary relations.

Since binary relations are the most common ones used in mathematics, mathematicians often omit the word binary and use the word "relation" when they mean "binary relation". Since we only discuss binary relations in this chapter we will sometimes omit the word binary.

9.2.2 Comparing relations

Equality of relations. Two relations R and Q are equal if $\mathbf{Source}(R) = \mathbf{Source}(Q)$, $\mathbf{Target}(R) = \mathbf{Target}(Q)$ and $\mathbf{pairs}(R) = \mathbf{pairs}(Q)$.

So to prove that two relations are equal you need to first show that their sources are the same and their targets are the same, and then show that they have the same pair-set. In practice it is almost always clear from the definitions of R and Q whether they have the same source and target. In that case when we prove $Q = R$, we only have to prove $\mathbf{pairs}(Q) = \mathbf{pairs}(R)$.

Containment of relations For relations R and Q, we say $Q \subseteq R$ and say R is a **subrelation** of R if $\mathbf{Source}(Q) \subseteq \mathbf{Source}(R)$, $\mathbf{Target}(Q) \subseteq \mathbf{Target}(R)$ and $\mathbf{pairs}(Q) \subseteq \mathbf{pairs}(R)$.

There are two important special cases of subrelations:

spanning subrelation and extensions A **spanning subrelation** of R is a subrelation Q with the same source and target as R. We also say that R is an **extension** of Q.

induced subrelations or restrictions If R is a relation on $S \times T$ and $S' \subseteq S$ and $T' \subseteq T$, then the **restriction of R to $S' \times T'$** is the relation $R[S', T']$ on $S' \times T'$ whose pair-set is the set of all pairs of $S' \times T'$ that belong to **pairs**(R). This is also called the **subrelation induced on $S' \times T'$**.

For homogeneous relations, for S' a subset of the ground set S, $R[S']$ is the **restriction of R to S'** or the **subrelation induced on S'**.

Example 9.19. 1. If we restrict the usual $\leq$ relation on $\mathbb{R}$ to the set $\mathbb{N}$ we get the usual $\leq$ relation on $\mathbb{N}$.

2. The divisibility relation on $\mathbb{N}$ is a spanning subrelation of the $\leq$ relation on $\mathbb{N}$.

3. Let Q be the relation on $\mathbb{R}$ with **pairs**$(Q) = \{(x, y) : |x - y| \leq 2\}$ and R be the relation on $\mathbb{R}$ with **pairs**$(Q) = \{(x, y) : |x - y| \leq 1\}$. Then R is a spanning subrelation of Q.

9.3 Operations on relations

Here we describe some important operations that produce a new relation from a single relation, or produce a new relation from two or more relations.

Definition 9.20. Let R be a relation on $S \times T$.

- The **reverse relation** $\overleftarrow{R}$ is the relation obtained from R by exchanging the source and target, and flipping every pair. More formally, **Source**$(\overleftarrow{R})$ = **Target**(R), **Target**$(\overleftarrow{R})$ = **Source**(R) and **pairs**$(\overleftarrow{R}) = \{(t, s) : (s, t) \in$ **pairs**$(R)\}$.

- The **complementary relation** R^c is the relation on $S \times T$ with relation set $S \times T \setminus$ **pairs**(R). **Source**(R^c) = **Source**(R) and **Target**(R^c) = **Target**(R) Thus for $(s, t) \in$ **Source**$(R) \times$ **Target**(R) we have sR^ct is and only if $s\not{R}t$.

- For $X \subseteq S$, $Y \subseteq T$, the **restriction of R to $X \times Y$** denoted $R[X,Y]$ is the subrelation of R on $X \times Y$ with pair-set $\mathbf{pairs}(R) \cap (X \times Y)$. If R is a homogeneous relation on S and $X \subseteq S$, the **restriction of R to X** is the subrelation R on X with pair set $\mathbf{pairs}(R) \cap (X \times X)$.

Next we define some ways to combine two binary relations to get a third.

Definition 9.21. Let Q and R be binary relations.

- The relation $Q \cup R$ has $\mathbf{Source}(Q \cup R) = \mathbf{Source}(Q) \cup \mathbf{Source}(R)$, $\mathbf{Target}(Q \cup R) = \mathbf{Target}(Q) \cup \mathbf{Target}(R)$ and $\mathbf{pairs}(Q \cup R) = \mathbf{pairs}(Q) \cup \mathbf{pairs}(R)$.

- The relation $Q \cap R$ has $\mathbf{Source}(Q \cap R) = \mathbf{Source}(Q) \cap \mathbf{Source}(R)$, $\mathbf{Target}(Q \cap R) = \mathbf{Target}(Q) \cap \mathbf{Target}(R)$ and $\mathbf{pairs}(Q \cap R) = \mathbf{pairs}(Q) \cap \mathbf{pairs}(R)$.

- The difference relation $Q \backslash R$ has the same source and target as Q, and $\mathbf{pairs}(Q \backslash R) = \mathbf{pairs}(Q) \backslash \mathbf{pairs}(R)$.

- The **composition of Q with R**, $Q \circ R$ is the relation with $\mathbf{Source}(Q \circ R) = \mathbf{Source}(R)$ and $\mathbf{Target}(Q \circ R) = \mathbf{Target}(Q)$ and $\mathbf{pairs}(Q \circ R) = \{(x,y) \in \mathbf{Source}(R) \times \mathbf{Target}(Q) : \exists z, xRz \wedge zRy\}$

Remark 9.22. The definition of composition of relations is closely connected to the previously defined composition of functions. In Example 9.15 we associated every function f to a relation R_f. If $f : C \longrightarrow D$ and $g : A \longrightarrow B$ are functions with target, then $f \circ g : A \longrightarrow D$. Then the relation $R_{f \circ g}$ associated to the function $f \circ g$ is equal to the composition of the relations $R_f \circ R_g$. See Exercise 9.3.3.

Exercise 9.3.1. Let R and Q be the relation defined on $S \times \mathcal{P}(S)$ where $S = \{1,2,3\}$, with $\mathbf{pairs}(R) = \{(s,T) : s \in T\}$ and $\mathbf{pairs}(Q) = \{(s,T) : 3 - s \in T\}$.

a. Find all members of $\mathbf{pairs}(\overleftarrow{R})$.

b. Find all members of $\mathbf{pairs}(R^c)$.

c. Find all members of $\mathbf{pairs}(R \cap Q)$

d. Find all members of $\mathbf{pairs}(R \cup Q)$

e. Find all members of $\mathbf{pairs}(R \setminus Q)$

Exercise 9.3.2.

a. Give an example of two relations R and Q for which $R \setminus Q \neq R \cap Q^c$.

b. Prove that if $\mathbf{Source}(R) \times \mathbf{Target}(R) \subseteq \mathbf{Source}(Q) \times \mathbf{Target}(Q)$ then $R \setminus Q = R \cap Q^c$.

c. Prove that if $\mathbf{Source}(R) \times \mathbf{Target}(R)$ is not a subset of $\mathbf{Source}(Q) \times \mathbf{Target}(Q)$ then $R \setminus Q \neq R \cap Q^c$.

Exercise 9.3.3. Let $g : A \longrightarrow B$ and $f : B \longrightarrow C$. Prove that $R_{f \circ g} = R_f \circ R_g$.

9.4 Properties of Homogenous Relations

For the remainder of this chapter we focus on relations (and the corresponding relationships) that are homogeneous, that is, whose source and target are the same.

Definition 9.23. One of the simplest relationships on a set A is the *equality relationship*. The associated relation is called the **diagonal relation on A** denoted $D(A)$. The pair set of this relation is $\{(a, a) : a \in A\}$.

Homogeneous binary relations (and their corresponding relationships) are classified according to some basic properties.

Definition 9.24. For a set A and relation R on A, we say that R (and its associated relationship) is

reflexive provided that for all $a \in A$, aRa.

antireflexive provided that for all $a \in A$, $\neg(aRa)$.

symmetric provided that for any $a, b \in A$, aRb implies bRa.

antisymmetric provided that for any $a, b \in A$ if $a \neq b$ and aRb then $b \not{R} a$. There is an alternate form of this definition: R is antisymmetric provided that for all $a, b \in A$ if aRb and bRa then $a = b$. In Exercise 9.4.1 you are asked to show that this alternate form is equivalent to the given definition. When proving that a relation is antisymmetric, the alternate form is often more convenient to use.

transitive provided that for any $a, b, c, \in A$, if aRb and bRc then aRc.

full provided that for any $a, b \in R$ with $a \neq b$, aRb or bRa. (This is not standard terminology.)

Exercise 9.4.1. Prove that the relation R is antisymmetric if and only if for all $a, b \in A$ if aRb and bRa then $a = b$.

Exercise 9.4.2. For each of the relationships described in Examples 9.3, 9.4, and 9.5 consider the associated relation, and determine which of the following properties hold: reflexive, anti-reflexive, symmetric, anti-symmetric, transitive, full

Exercise 9.4.3.

1. Give an example of a relation that is neither reflexive, nor anti-reflexive. (Thus reflexive and anti-reflexive are not negations of each other.)

2. Give an example of a relation that is neither symmetric, nor anti-symmetric. (Thus symmetric and anti-symmetric are not negations of each other.)

3. Are there any relations that are both reflexive and anti-reflexive? Are there any that are both symmetric and anti-symmetric?

Exercise 9.4.4. Suppose R is a transitive relation on A.

1. Prove: For any $x, y \in A$ if xRy then $R(y) \subseteq R(x)$.

2. Prove or give a counterexample: For any $x, y \in A$ if $R(y) \subseteq R(x)$ then xRy.

Exercise 9.4.5.

1. Prove that for any relation R on A, R is reflexive if and only if $\mathbf{pairs}(D_A) \subseteq \mathbf{pairs}(R)$.

2. Formulate and prove a similar statement that says when R is anti-reflexive.

Exercise 9.4.6.

1. Prove that a relation R on A is symmetric if and only if $R = \overleftarrow{R}$.

2. Formulate and prove a similar statement that says when R is anti-symmetric.

There are two types of relations that arise frequently in mathematics called *partial order relations* and *equivalence relations*. In the next following sections we discuss these in detail.

9.5 Equivalence relations

In many situations we classify objects of a particular set into groups of "similar" objects. We might classify real numbers into three types "positive", "negative" and "zero", In geometry we classify triangles according to their sequence of angles (such as 30,60,90 triangles) and two triangles with the same sequence of angles are said to be similar. We can classify finite sets according to the number of elements they have; we say that two sets are **equal-sized** if they have the same number of elements.

This classification is formalized mathematically as a partition of the set of objects. Recall that a **partition** of the set X is a set $\mathcal{P}$ such that (i) every member of $\mathcal{P}$ is a nonempty subsets of X, (ii) for all $x \in X$ there is a member $A \in \mathcal{P}$ such that $x \in A$, and (iii) For all $A, B \in \mathcal{P}$ if $A \cap B \neq \emptyset$ then $A = B$. In words, a partition consists of disjoint nonempty subsets of X whose union is X.

Definition 9.25. Equivalence relations. Suppose X is a set and let $\mathcal{P}$ be a partition of X. The **equivalence relation** $R_{\mathcal{P}}$ *induced by the partition* $\mathcal{P}$ is the set of pairs $(x, y) \in X \times X$ such that x and y are in the same part of $\mathcal{P}$. The parts of the partition are called the **equivalence classes** of the equivalence relation.

Equivalence relations have a very simple structure: once you know the partition it's easy to see which elements are related to each other.

Example 9.26.

1. Take the set of all lists with entries from some universe $\mathbb{N}$. Every list has an associated set of the entries that appear. For example, the set associated to the list $(1, 3, 5, 3, 6, 3, 1)$ is $\{1, 3, 5, 6\}$. If we classify lists according to the associated set of entries we get an equivalence relation on lists.

2. We can also classify lists according to their lengths so that all lists with the same length are in the same equivalence class.

3. Consider what happens when you round a real number to the closest integer. This rounding rule is ambiguous for numbers like 3.5 since it is equally close to 3 and 4; in this case let's say we round up to the nearest integer. Let's say that two real numbers are *approximately equal* if they round to the same integer, so 8.8 and 9.3 are approximately equal. The relationship *approximately equal* defines an equivalence relation.

4. Let's classify integers according to the largest power of 2 they are divisible by. Thus 24 and 40 are divisible by 8 but not 16 and are classified the same.

Exercise 9.5.1. Precisely describe the partition into equivalence classes for the approximately equal relation.

If $\mathcal{P}$ is a partition of X, the members of $\mathcal{P}$ are called the **$\mathcal{P}$-classes**. Every $x \in X$ belongs to a unique $\mathcal{P}$-class called the **$\mathcal{P}$-class of x** denoted $[x]_{\mathcal{P}}$.

System of representatives for a partition. Suppose $\mathcal{P}$ is a partition of set X. A **system of representatives** for $\mathcal{P}$ is a set S that intersects each member of $\mathcal{P}$ in a single element. If we fix a system of representatives S, then for each $P \in \mathcal{P}$ there is a unique $x \in S$ such that $P = [x]_{\mathcal{P}}$.

Constructing a system of representatives is essentially the same as providing a **representative selection function**. This is a function $r : \mathcal{P} \longrightarrow X$ with the property that $r(B) \in B$ for all $B \in \mathcal{P}$. The range of a representative selection function is a system of representatives.

Exercise 9.5.2. Suppose X is a set and $\mathcal{P}$ is a partition of X.

1. Prove: For any representative selection function r for $\mathcal{P}$, $\mathbf{Range}(r)$ is a system of representatives.

2. Prove: For any system of represenatives S of $\mathcal{P}$, there is a unique representative selection function for P whose range is S.

A system of representatives provides a convenient way to refer to the parts of a given partition. The choice of the system of representatives is arbitrary, but it is usually most useful to choose the system of represntatives to be as simple as possible.

Example 9.27. In Example 9.26:

- For Example 1, the set of all lists of natural numbers whose entries are in strictly increasing order is a system of representatives.

- For Example 2, the set of all lists whose only entries are 1 is a system of representatives

- For Example 3, the set of integers is a system of representatives.

- For Example 4, the set $\{2^j : j \in \mathbb{Z}_{\geq 0}\}$ is a system of representatives.

Exercise 9.5.3. For each of the examples in Example 9.27 explain why the given set is a system of representatives for the associate partition.

Theorem 9.28. *Every equivalence relation is reflexive, symmetric and transitive.*

Exercise 9.5.4. Prove Theorem 9.28: Every equivalence relation is reflexive, symmetric and transitive.

Remarkably it turns out that the converse of Theorem 9.28 is true. That is, any homogeneous relation that is reflexive, symmetric and transitive, must be an equivalence relation for some partition of the underlying set.

Theorem 9.29. *Suppose R is a relation on X that is transitive, symmetric, and reflexive. Then the set of sets $\mathcal{P} = \{R(x) : x \in X\}$ is a partition of X and $R = R_{\mathcal{P}}$.*

Remark 9.30. We defined an equivalence relation to be a relation that comes from a partition by saying two objects are related if they belong to the same part of the partition. Another way to define equivalence relation (which appears in many other books) is to say that an equivalence relation is a relation that is transitive, symmetric and reflexive. We refer to the first definition as the *partition-based* definition of equivalence relations and the second definition as the *property-based* definition. Theorems 9.28 and 9.29 together imply the remarkable fact that these two very different definitions describe exactly the same type of relation.

Proof. Suppose R is an equivalence relation on X and let $\mathcal{P} = \{R(x) : x \in X\}$. We claim that $\mathcal{P}$ is a partition of X and that $R = R_{\mathcal{P}}$.

First we prove that $\mathcal{P}$ is a partition of X. We must show that (i) Every member of $\mathcal{P}$ is a nonempty subset of X, (ii) for all $x \in X$, there is a member of $\mathcal{P}$ that has x as a member, and (iii) for all $A, B \in \mathcal{P}$ if $A \cap B \neq \emptyset$ then $A = B$.

Proof of (i). Suppose $S \in \mathcal{P}$. Then there is a member of X we'll call x such that $S = R(x)$. Since R is a homogeneous relation on X, $R(x) \subseteq X$ and since R is reflexive, $x \in R(x)$ and so $R(x)$ is nonempty.

Proof of (ii). Suppose $x \in X$ is arbitrary. Then $R(x) \in \mathcal{P}$. Since R is reflexive, $x \in R(x)$.

Proof of (iii). Suppose $A, B \in \mathcal{P}$. Assume that $A \cap B \neq \emptyset$. We must show $A = B$. For this we must show $A \subseteq B$ and $B \subseteq A$. By definition of $\mathcal{P}$ there are objects we'll call $a, b \in X$ such that $A = R(a)$ and $B = R(b)$.

Proof that $R(a) \subseteq R(b)$. Let x be an arbitrary member of $R(a)$. We must show $x \in R(b)$, which means we must show bRx. By definition of $R(a)$, we have aRx. Since $R(a) \cap R(b) \neq \emptyset$ there is an object we'll call y such that aRy and bRy. By symmetry of R we have yRa and by transitivity bRy and yRa implies bRa. Since bRa and aRx we have bRx by transitivity, as required to show $A \subseteq B$.

The proof that $R(b) \subseteq R(a)$ is analogous by interchanging a and b in the above proof.

So we've shown that $\mathcal{P}$ is a partition. It remains to show that $R = R_{\mathcal{P}}$. For this we must show that for all $x, y \in X$ xRy if and only if $xR_{\mathcal{P}}y$. Suppose $x, y \in X$. We must prove two things: (a) if xRy then $xR_{\mathcal{P}}y$ and (b) if $xR_{\mathcal{P}}y$ then xRy.

Part (a). Assume xRy. To show $xR_{\mathcal{P}}y$ we need that x and y belong to the same part of $\mathcal{P}$. By definition of $\mathcal{P}$, $R(x) \in \mathcal{P}$ and $x \in R(x)$ by reflexivity and $y \in R(x)$ by assumption.

Part (b). Assume $xR_{\mathcal{P}}y$. Since $\mathcal{P}$ is a partition, there is a unique set $S \in \mathcal{P}$ that contains x. By assumption and the definition of $R_{\mathcal{P}}$, $y \in S$. $R(x) \in \mathcal{P}$, by definition of $\mathcal{P}$ and by reflexivity $x \in R(x)$ and so $S \cap R(x) \neq \emptyset$. Therefore $S = R(x)$. Therefore $y \in R(x)$ and so xRy. □

Given an equivalence relation R the parts of the associated partition $\mathcal{P}_R$ are called the *equivalence classes* of R.

Remark 9.31. The theorem tells us that if R is an equivalence relation on X then the set $\{R(a) : a \in X\}$ is a partition of X. This does not mean that all of the sets $R(a)$ are different. What it does mean is that for any two members a, b of X we either have $R(a) = R(b)$ or $R(a) \cap R(b) = \emptyset$.

The importance of Theorem 9.29 is that it gives us a way to show that a relation is an equivalence relation without knowing the partition into equivalence classes: we just show that the relation is reflexive, symmetric and transitive. By studying the relation and using Theorem 9.29 we can figure out the equivalence classes.

9.5.1 Equivalence of integers modulo an integer k

We now discuss in detail some important equivalence relations on the set of integers.

In the following discussion think of k as an arbitrary but fixed integer. For integers i and j we say that **i is congruent to j modulo k**, or **i is congruent to j mod k**, written $i \equiv_k j$, if $i - j$ is divisible by k. This relationship is called the **congruence modulo** k relationship or simply the **mod** k relationship. Let us define C_k to be the associated relation on the set $\mathbb{Z}$. Thus for every integer j, $C_k(j) = \{i \in \mathbb{Z} : k|(i-j)\}$. Let $\mathcal{C}_k$ denote the set $\{C_k(j) : j \in \mathbb{Z}\}$.

Theorem 9.32. *For every positive integer k, the relation C_k is reflexive, symmetric and transitive.*

Exercise 9.5.5. Prove Theorem 9.32, that for every positive integer k, C_k is a reflexive, symmetric and transitive relation on $\mathbb{Z}$.

Example 9.33. When $k = 2$, two numbers are congruent modulo 2, if they differ by an even number. It is easy to see that there are exactly two equivalence classes, the even numbers and the odd numbers.

Given Theorem 9.32, we may apply 9.29 to conclude that $\mathcal{C}_k$ is a partition of $\mathbb{Z}$ and the relation C_k is the same as the equivalence relation $R_{\mathcal{C}_k}$. Since $\mathcal{C}_k = \{C_k(j) : j \in \mathbb{Z}\}$, this gives an infinite list of possible equivalence classes, but many of these equivalence classes are the same. We'd like to know: (1) Is the number of different equivalence classes finite (no matter what k is)? (2) If so, how many are there?

The following theorem tells us that for every positive integer k, there are exactly k equivalence classes, and that the set $\{0, \ldots, k-1\}$ is a system of representatives.

Theorem 9.34. *For any positive integer k,*

1. *The sets $C_k(j)$ for $j \in \{0, \ldots, k-1\}$ are all different.*
2. *Every integer belongs to one of the classes $C_k(0), \ldots, C_k(k-1)$*
3. *The partition of $\mathbb{Z}$ into equivalence classes modulo k is $\{C_k(0), \ldots, C_k(k-1)\}$.*

Proof	**Commentary**
Suppose k is an arbitrary positive integer. For simplicity we drop the subscript and write simply C instead of C_k.	

Proof of Part 1. Suppose i and j are arbitrary members of $\{0,\ldots,k-1\}$. We want to show that if $i \neq j$ then $C(i) \neq C(j)$. We prove the contrapositive. So assume $C(i) = C(j)$; We must show $i = j$. By definition of $C(i)$ we have $i \in C(i)$ and since $C(j) = C(i)$, we have $i \in C(j)$. Then $(i-j)/k$ is an integer, and so is $|i-j|/k$. Since i and j are between 0 and $k-1$, $0 \leq |i-j| \leq k-1$ and so $0 \leq |i-j|/k < 1$. The only integer in that interval is 0, which means $|i-j| = 0$ and so $i = j$, as required. We conclude that if $i \neq j$ then $C(i) \neq C(j)$.

The proof of the first part is straightforward. We want to show that no two of the sets are the same. The sets are indexed by the integers in $\{0,\ldots,k-1\}$ so we pick an arbitrary i and j from that set. We switch to proving the contrapositive which is often more convenient when proving something of the form "if $a \neq b$ then $x \neq y$" which becomes "if $x = y$ then $a = b$". In this case we assume $C(i) = C(j)$ which implies k is a divisor of $i-j$. When working with $i-j$, we use absolute value to avoid having to divide into cases depending on whether $i > j$ or $i \leq j$.

Proof of Part 2. Suppose that n is an arbitrary integer. We must show that there is a $j \in \{0,\ldots,k-1\}$ such that $n-j$ is divisible by k.

Our job is to give instructions to find an integer j so that $j < k$ and $j \geq 0$ and $n-j$ is divisible by k. Our strategy will be this: We'll ignore the first requirement and consider the set of all nonnegative integers m such that $n-m$ is divisible by k. To have any hope we need that this set is nonempty. Once we know the set is nonempty we can use the well-ordering principle (discussed prior to Proposition 7.4 to say that this set has a smallest member. We will then show that this smallest member must be less than k and therefore satisfies all the requirements.

Let W be the set of all nonnegative integers m such that $n-m$ is divisible by k. We must pick j from W. First we show that W is nonempty. We split into cases according to whether $n \geq 0$ or not.

Case a. Assume $n \geq 0$. Then $n \in W$.

We split into cases here because when $n \geq 0$ the argument that W is nonempty is very simple, but that argument does not work when $n < 0$. For $n < 0$ we need a different proof.

Case b. Assume $n < 0$. We have to show that there is a nonnegative integer t so that $n - t$ is divisible by k. We note that nk is divisible by k and less than or equal to n so we let $t = n - nk$. This is nonnegative since $nk \leq n$. Also $n - t = nk$ is divisible by k. So $t \in W$. and so W is nonempty.

The well-ordering principle for $\mathbb{N}$ says that any nonempty subset of $\mathbb{Z}_{\geq 0}$ has a smallest member. Since W is such a set, it has a smallest member, which we will call j. Clearly $j \geq 0$ since every member of W is nonnegative. We also claim that $j < k$. Suppose for contradiction that $j \geq k$. Let $j' = j - k$. We claim that $j' \in W$, which will contradict the selection of j to be the smallest member of W. To show that $j' \in W$ we need (1) $n - j'$ is divisible by k and (2) $j' \geq 0$.

Proof of (1): We have $(n - j')/k = (n - j + k)/k = (n - j)/k + 1$. Since $j \in W$ we have the $(n - j)/k$ is an integer and therefore $(n - j')/k$ is also an integer.

Proof of (2): Since $j \geq k$, $j' = j - k \geq k - k = 0$.

We have therefore concluded that $j' \in W$ which contradicts that j is the smallest member of W. Since assuming $j \geq k$ leads to a contradiction, we conclude $j < k$. So we have found a $j \in \{0, \ldots, k-1\}$ such that $n \in C(j)$ as required to complete the proof of Part 2.

Having picked the integer j, we now have to demonstrate that j has the required properties. We do this by contradiction. Using proof by contradiction is very common in proofs that use the well-ordering principle to select a smallest or largest member of a set of numbers.

Proof of Part 3. Since the sets $C(j)$ are equivalence classes for the relation C, we know from Theorem 9.29 that any two of them are either the same or disjoint. From the first part that the sets $C(0), \ldots, C(k-1)$ are all different, so they must be all disjoint. By the second part of the theorem, their union is all of $\mathbb{Z}$, and so they partition $\mathbb{Z}$. □

One important consequence of the above theorem is:

Theorem 9.35. *(Quotient-remainder theorem) Suppose k is an arbitrary positive integer. For every integer n there are is a unique integer q and a unique number $r \in \{0, \ldots, k-1\}$ such that $n = qk + r$.*

Remark 9.36. This theorem states a familiar fact taught in middle school, when you divide an integer n by a positive integer k you get a quotient q and a remainder r that is between 0 and $q - 1$.

Proof. Suppose k is a positive integer and suppose that n is an integer. By Theorem 9.34 the sets $\{C_k(j) : j \in \{0, 1, \ldots, k-1\}\}$ partition $\mathcal{Z}$. So there is an integer, which we will call r, such that $r \in \{0, 1, \ldots, k-1\}$ and $n \in C_k(r)$. Since $n \in C_r$, $n - r$ is a multiple of k. Letting $q = (n-r)/k$ we have $n = qk + r$ as required. □

9.5.2 Some additional examples of equivalence relations

Here are some additional examples of relations. In the exercise that follows you'll show that each is an equivalence relation.

Example 9.37. Some additional examples of relations that are reflexive, symmetric and transitive (and are therefore equivalence relations).

1. Real Numbers under rational equivalence xRy if there are integers a, b such that $ax = by$.

2. Integers under odd multiple equivalence: xRy if there are odd integers a, b such that $ax = by$.

3. Integers under *square equivalence* xRy: if there are integers a, b such that $xa^2 = yb^2$.

4. For $S \subseteq \mathbb{R}$, we define a relation N_S on S by the rule xN_Sy if the interval $I[x, y] \subseteq S$ (where $I[x, y] = [\min(x, y), \max(x, y)]$.)

5. Consider the relation W on the ground set set $\mathbb{R}^2$, with $(a, b)W(c, d)$ if and only if $ad = bc$. (Note: This may be confusing at first. Here the ground set is a set of ordered pairs, so pairs in the relation are ordered pairs, each of whose coordinates is an ordered pair.)

Exercise 9.5.6. For each relation in Example 9.37, prove that the relation is reflexive, symmetric and transitive.

Exercise 9.5.7. For parts 2,3 and 5 of Example 9.37 determine a simple description of the equivalence classes and prove that your description is correct.

Exercise 9.5.8. Let R be a relation on X. Let us say that two elements x and y are R-twins if for every $z \in X$ we have xRz if and only if yRz and zRx if and only if zRy. Let $T = T(R)$ be the relation on X where xTy if x and y are R-twins.

1. Prove that T is an equivalence relation on X.

2. Give an example of a reflexive relation R on a 6 element set for which $T(R)$ has two equivalence classes of size 3.

3. Suppose R is the divisibility relation on $\mathbb{Z}$. What is $T(R)$?

9.5.3 The components of a relations

Suppose R is a homogeneous relation on A and suppose $w = (w_0, w_1, \ldots, w_k)$ is a list of members of A (possibly with the same member appearing more than once.)

- We say that w is an **R-walk** provided that for all $i \in \{1, \ldots, k\}$, $w_{i-1}Rw_k$. The walk w is said to **start at w_0** and **end at w_k**. The pair (w_{i-1}, w_i) is the **ith step** of the walk. The number of steps of the walk is therefore k which is one less than the length of the list w.

- The R-walk $w = (w_0)$ consisting of a one entry list is called a **trivial walk**. A trivial walk has no steps. A walk with at least two entries (and therefore at least one step) is a **non-trivial walk**.

- An R-walk w is a **closed walk** if it is non-trivial and $w_k = w_0$.
- An R-walk w is a **R-path** provided that $w_0, \ldots, w_k$ are all distinct.
- An **R-cycle** w is a closed walk such that $w_1, \ldots, w_k$ are all distinct. Thus the only two equal entries of a closed walk are w_0 and w_k.
- For $x, y \in A$, we say that **y is R-reachable from x** provided that there is an R-walk from x to y.

Proposition 9.38. *For any $x, y \in A$, y is R-reachable from x, if and only if there is an R-path from x to y.*

Exercise 9.5.9. Prove Proposition 9.38: Suppose R is a homogeneous relation on A. For any $x, y \in A$, y is R-reachable from x, if and only if there is an R-path from x to y.

Definition 9.39. For any relation R on set A

- Define the relation R^* to be the relation on A such that $\mathbf{pairs}(R^*) = \{(x, y) \in A^2 : y$ is R-reachable from $x\}$. R^* is called the **transitive closure of R**.
- Define the relation $\bar{R}$ to be the relation on A such that $\mathbf{pairs}(\bar{R}) = \{(x, y) \in A^2 : y$ is R-reachable from x and x is R-reachable from $y\}$.

In all of the propositions and theorems below, A is an arbitrary set and R is an arbitrary homogeneous relation on A.

Proposition 9.40. *For any relation R,*

1. *R^* is reflexive and transitive.*
2. *If Q is any extension of R (meaning $\mathbf{pairs}(R) \subseteq \mathbf{pairs}(Q)$ that is transitive and reflexive, then Q is an extension of R^*.*

Proposition 9.41. 1. *R is a spanning subrelation of R^*.*

2. *$R = R^*$ if and only if R is transitive and reflexive.*

Proposition 9.42. *$\bar{R}$ is an equivalence relation on A.*

The equivalence classes of $\bar{R}$ are called the **strong components of A**.

Exercise 9.5.10. Prove Proposition 9.40: For any homogeneous relation R on A, R^* is reflexive and transitive.

Exercise 9.5.11. Prove Proposition 9.42: Suppose R is a homogeneous relation on A. Then $\bar{R}$ is an equivalence relation on A.

Exercise 9.5.12. Suppose R is a homogeneous relation on A.

1. Prove or disprove: If R is symmetric then R^* is symmetric.

2. Prove or disprove: If R is antisymmetric then R^* is antisymmetric.

9.6 Partial order relationships and relations

A partial order relationship on a set is a way of comparing members where we say that one member x is, in some sense, "smaller than or equal to" another. The most familiar example is numbers where "smaller than or equal to" means "$\leq$". Here are some other examples:

Example 9.43.

1. $\mathcal{P}(U)$ **with the relationship** $\subseteq$ (Recall that $\mathcal{P}(U)$ is the set of subsets of U.)

2. $\mathbb{N}$ **with the relationship "is a divisor of".** Recall that we say that m is a divisor of n, written $m|n$ provided that n/m is an integer.

3. **The set** A^* **with the relationship "is a prefix of".** Recall that for a set A, A^* is the set of lists with entries in A. For lists a, b we say that a is a **prefix** of b, denoted $a \leq_{\text{pre}} b$ provided that $\textbf{length}(a) \leq \textbf{length}(b)$ and for all $i \in \{1, \ldots, \textbf{length}(a)\}$, $b_i = a_i$.

Exercise 9.6.1.

Prove that the each of the relationships in Example 9.43 satisfy the reflexive, anti-symmetric and transitive properties.

With these examples in mind, we make the following definition

Definition 9.44. A relation R on X is a **partial order** on X provided that it is **anti-symmetric** and **transitive**, and **reflexive**. If R is a partial order, then we also say that the associated relationship is a partial order.

Exercise 9.6.2. Consider the following relations on $\mathcal{P}_{\textbf{fin}}(\mathbb{Z})$

1. The relation consisting of pairs (X, Y) such that $|X| = |Y|$.

2. The relation consisting of pairs (X, Y) such that $|X| \leq |Y|$.

3. The relation consisting of pairs (X, Y) such that either $X = Y$ or $Y \setminus X$ is nonempty and every member of $Y \setminus X$ is greater than every member of X.

For each of these relations decide whether it is a partial order or not, and provide a proof of your answer.

Notation for partial orders When we are discussing a partial order relation R on a set A,

- $x \leq_R y$ means xRy,
- $x <_R y$ means xRy and $x \neq y$,

If the partial order relation R is clear from context we may omit the subscript "R" and write simply $x \leq y$ and $x < y$. We can do this provided that that we are careful that:

- In the present discussion, there is only one partial order relation mentioned.
- There is no possibility that the reader will be confuse this with the usual meaning of $\leq$ for numbers.

If R is a partial order on X, then two members x and y of X are said to be **comparable** if $x \leq_R y$ or $y \leq_R x$ and said to be **incomparable** if neither $x \leq_R y$ nor $y \leq_R x$.

Definition 9.45. A partial order on A with the property that any two members of A are comparable is called a **total order** or **linear order** on A. A total order is sometimes also called a **chain**.

For example, the $\leq$ relation on $\mathbb{R}$ is a total order.

Exercise 9.6.3. Show that the partial order relations described in Example 9.43 other then $\mathbb{R}$ with the usual $\leq$ relationship are not total orders.

Strict Partial Orders Our definition of a partial order R on X requires that the relation be reflexive, which means that xRx for all $x \in X$. Above, we defined the notation $x \leq_R y$ to mean xRy and $x <_R y$ to mean that xRy and $x \neq y$.

The relationship $x <_R y$ defines another relation that is closely related to R but slightly different. The pairs in this relation are the pairs of R with the exception of the pairs (x, x) for $x \in X$. Therefore this relation is equal to the relation $R \backslash D(X)$ where $D(X)$ is the diagonal relation defined earlier, whose pair set is $\{(x, x) : x \in X\}$. We refer to this relation as $R \backslash D$.

Notice that a relation of the form $R \backslash D$ where R is a partial order does not meet the definition of a partial order because a partial order is reflexive. This leads us to define a related concept:

Definition 9.46. A relation R on a set X is a **strict partial order** provided that it is antireflexive, antisymmetric, and transitive.

So a strict partial order differs from a partial order in that the former is antireflexive while the latter is reflexive.

There is a natural correspondence between strict partial orders and partial orders.

Proposition 9.47. 1. *For any partial order relation R on A, we have $\boldsymbol{pairs}(D(A)) \subseteq \boldsymbol{pairs}(R)$ and the relation $R \backslash D$ on A with $\boldsymbol{pairs}(R \backslash D) = \boldsymbol{pairs}(R) \backslash \boldsymbol{pairs}(D(A))$ is a strict partial order.*

2. *For any strict partial order relation Q on A, we have $\boldsymbol{pairs}(D(A))$ is disjoint from $\boldsymbol{pairs}(Q)$ and the relation $Q \cup D(A)$ on A with $\boldsymbol{pairs}(Q \cup D(A)) = \boldsymbol{pairs}(Q) \cup \boldsymbol{pairs}(D(A))$ is a partial order.*

Proof. We prove Part 1, and leave Part 2 as an exercise.

Proof of Part 1. Suppose A is an arbitrary set and R is an arbitrary partial order on A. We first show that $\mathbf{pairs}(D(A)) \subseteq \mathbf{pairs}(R)$. Each pair in $\mathbf{pairs}(D(A))$ has the form (a, a) where $a \in A$. Suppose a is an arbitrary member of A, then since R is reflexive $(a, a) \in R$ and so $\mathbf{pairs}(D(A)) \subseteq \mathbf{pairs}(R)$.

Now let $R \backslash D$ be the relation on A with $\mathbf{pairs}(R \backslash D) = \mathbf{pairs}(R) \backslash \mathbf{pairs}(D(A))$. We must show that $R \backslash D$ is a strict partial order, so we must show that $R \backslash D$ is anti-reflexive, anti-symmetric and transitive.

Proof that $R \backslash D$ is anti-reflexive. Suppose $a \in A$ is arbitrary. We must show $(a, a) \notin \mathbf{pairs}(R \backslash D)$. We have $(a, a) \in D(A)$ and by the definition of $\mathbf{pairs}(R \backslash D)$, $(a, a) \notin \mathbf{pairs}(R \backslash D)$.

Proof that $R \backslash D$ is anti-symmetric. Suppose x and y are arbitrary members of A we have to prove that $x(R \backslash D)y$ and $x \neq y$ implies $\neg y(R \backslash D)x$. We'll actually prove something stronger: if $x(R \backslash D)y$ then $\neg y(R \backslash D)x$. (**Question**: *Why is it enough to prove this?*) Assume for contradiction that $x(R \backslash D)y$ and $y(R \backslash D)x$. Since $\mathbf{pairs}(R \backslash D) \subseteq \mathbf{pairs}(R)$ we have xRy and yRx and since R is anti-symmetric, $x = y$. But then $x(R \backslash D)x$ which is impossible by the definition of $R \backslash D$.

Proof that $R\backslash D$ is transitive. Suppose x,y,z belong to A. Assume $x(R\backslash D)y$ and $y(R\backslash D)z$. We must show that $x(R\backslash D)z$, which is equivalent to xRy and $x \neq y$.

Proof that $x(R\backslash D)z$: Since $\mathbf{pairs}(R\backslash D) \subseteq \mathbf{pairs}(R)$, we have xRy and yRz and therefore xRz.

Proof that $x \neq z$. Suppose for contradiction that $x = z$. Then since yRz we have yRx. Since yRx and xRy, anti-symmetry of R implies $x = y$, which contradicts $x(R\backslash D)y$. Therefore $x \neq z$.

This completes the proof that $R\backslash D$ is transitive. □

Exercise 9.6.4. Prove the second part of Proposition 9.47:For any strict partial order relation Q on A, we have $\mathbf{pairs}(D(A))$ is disjoint from $\mathbf{pairs}(Q)$ and the relation $Q \cup D(A)$ on A with $\mathbf{pairs}(Q \cup D(A)) = \mathbf{pairs}(Q) \cup \mathbf{pairs}(D(A))$ is a partial order.

Introducing a partial order When we state and prove universal principles about partial order relations, or make general defintions pertaining to partial order relations we'll need to introduce an arbitrary partial order relation. We have to be a little careful about this because a partial order relation involves both a set A and the relation R, and also we may want to introduce a special symbol such as $\preceq$ to represent the less than or equal relation in the relation.

If A is a specific set (such as the set of integers), or A is an unspecified set that was previously introduced into the scenario, we might write one of the following:

- "Suppose R is a partial order on A."
- "Suppose R is a partial order on A with relation symbol $\preceq$. " Here besides introduing R, we are also saying that the notation $x \preceq y$ will represent that $(x, y) \in R$. (The particular choice of "$\preceq$" is arbitrary; we are free to choose another symbol as long as we tell the reader. It is conventional to use a symbol such as $\preceq$ that has a horizontal bar at the bottom for partial orders and the same symbol without the horizontal bar $\prec$ for the associated strict order.

Often, the set A and the set R are introduced at the same time. We might write:

- "Suppose A is an arbitrary set and suppose R is a partial order relation on A with relation symbol $\preceq$."
- "Suppose R is a partial order on A with relation symbol $\prec$. "

In the first introduction, we first introduce A and then introduce R to be a partial order on A. In the second shorter introduction, the set A is introduced by implication. This is similar to when we introduce a function by saying: "Suppose $f : A \longrightarrow B$ is an arbitrary function" which may serve to introduce the two sets A and B and the function f. The interpretation of this sentence depends on whether A was already active. If it was already active, then we are simply introducing R. If A was not already active we are introducing both R and A.

There is yet another form for introducing a partially ordered set. We might just say

> Suppose A is an arbitrary partially ordered set with relation symbol $\preceq$.

The term **partially ordered set** (abbrevatied **poset**) means a set that has a partial order relation associated to it. Here we introduced the set A, and the partial order relation on A, but rather than give a name to the relation, we announce that the notation $x \preceq y$ will denote that (x, y) belongs to the partial order relation.

We might shorten this further and say simply:

> Suppose A is an arbitrary partially ordered set.

When we use this form, we don't specify the name of the relation or the relationship symbol. In this case, when we want to say that (x, y) is a member of the relation, we write $x \leq_A y$. We can omit the subscript A and write $x \leq y$ provided that there is no risk in confusing this relationship with another partial order.

9.6.1 Upper bounds, lower bounds, minimum, maximum, minimal and maximal

In this section we introduce some important terminology related to subsets of a partially ordered set. The terms are:

- lower bound
- upper bound
- minimum
- maximum

- minimal member
- maximal member

Throughout this section A denotes a fixed partially ordered set. We use the symbols $\leq_A$ to denote the relation symbol for the partial order, and define $<_A$, $\geq_A$ and $>_A$ in the obvious way.

Here are the definitions of the first four terms

Definition 9.48. For a partially ordered set A, a subset S of A, and member $z \in A$:

- z is a **lower bound** of S, or S is **bounded below** by z, provided that for all $s \in S$, $z \leq s$. (It is not necessary that z be a member of S)
- We say that S is **bounded below** provided that there is at least one lower bound for S.
- z is an **upper bound** of S, or S is **bounded above** by z provided that for all $s \in S$, $z \geq s$.
- We say that S is **bounded above** provided that there is at least one lower bound for S.
- z is a **minimum of S** provided that z is a lower bound of S and $z \in S$.
- z is a **maximum of S** provided that z is an upper bound for S and $z \in S$.

These terms are probably familiar to you in the context of sets of real numbers with the standard $\leq$ order.

Example 9.49. Consider the following sets of numbers with the standard $\leq$ order.

- For the set $\{2, 4, 6, 8\}$, every number in the interval $(-\infty, 2]$ is a lower bound, every number in $[8, \infty)$ is an upper bound, 2 is a minimum and 8 is a maximum.
- For the set $\mathbb{R}_{>0}$ of positive real numbers, any number in $(-\infty, 0]$ is a lower bound, there are no upper bounds, and there is no minimum or maximum.

- For the set $[1, 3)$, every number in $(-\infty, 1]$ is a lower bound, every number in $[3, \infty)$ is an upper bound, 1 is a minimum and there is no maximum. (Why not?)

Example 9.50. Consider the set $\mathcal{P}(\mathbb{N})$ with the subset order. The set $\{\{1,2,3,4\},\{1,2,4,5\},\{1,2,3,4,5,6\}\}$ has maximum $\{1,2,3,4,5,6\}$ but has no minimum. An element $S \in \mathcal{P}(\mathbb{N})$ is an upper bound if it is a superset of $\{1,2,3,4,5,6\}$ and is a lower bound if it is a subset of $\{1,2\}$.

Theorem 9.51. *Let S be a set and consider the powerset $\mathcal{P}(S)$ as a partially ordered set under the inclusion order. Then for any $\mathcal{B} \subseteq \mathcal{P}(S)$:*

1. *A subset X of S is an upper bound of $\mathcal{B}$ if and only if $\bigcup_{B\in\mathcal{B}} B \subseteq X$.*

2. *A subset X of S is a lower bound of $\mathcal{B}$ if and only if $X \subseteq \bigcap_{B\in\mathcal{B}} B$.*

Proof. For Part 1, suppose X is an arbitrary subset of S. We must prove: (a) X is an upper bound of $\mathcal{B}$ implies $\bigcup_{B\in\mathcal{B}} B \subseteq X$, and (b) if $\bigcup_{B\in\mathcal{B}} B \subseteq X$ then X is an upper bound of $\mathcal{B}$.

To prove (1a) assume X is an upper bound of $\mathcal{B}$. We must show $\bigcup_{B\in\mathcal{B}} B \subseteq X$. Suppose z is an arbitrary member of $\bigcup_{B\in\mathcal{B}} B$. We must show $z \in X$. Since $z \in \bigcup_{B\in\mathcal{B}} B$, there is a member C of $\mathcal{B}$ such that $z \in C$. Since X is an upper bound for $\mathcal{B}$ and $C \in \mathcal{B}$ we have $C \subseteq X$ and so $z \in C$ implies $z \in X$. Thus we've shown $UB(\mathcal{B}) \subseteq [\bigcup_{B\in\mathcal{B}} B, \infty)$.

To prove (1b) suppose $\bigcup_{B\in\mathcal{B}} B \subseteq X$. We must show X is an upper bound of $\mathcal{B}$. Suppose C is an arbitrary member of $\mathcal{B}$; we must show $C \subseteq X$. Since $C \in \mathcal{B}$, $C \subseteq \bigcup_{B\in\mathcal{B}} B$ and this is a subset of X. By transitivity of $\subseteq$, we conclude that $C \subseteq X$.

The proof of the second equation is left as an exercise. □

Exercise 9.6.5. Prove Part 2 of Theorem 9.51.

Exercise 9.6.6. Consider the divisibility partial order on the natural numbers.

1. Provide a lower bound and an upper bound for the set $\{6, 15, 21\}$

2. For each k, provide an upper bound for the set $\{1, \ldots, k\}$.

From Example 9.49 we see that a subset of a poset may or may not have any lower bounds or upper bounds and may not have a minimum element or a maximum element. Intuitively, if S has a minimum, the minimum plays the role of a *smallest* member of S, and if S has a maximum, the maximum plays the role of a *largest* member of S.

Here are two additional important definitions for partially ordered sets:

Definition 9.52. For a partially ordered set A, subset S of A and $z \in A$:

- z is a **maximal member of S** or is **maximal in S** provided that there is no member y of S for which $z >_A y$.

- z is a **minimal member of S** or is **minimal in S** provided that there is no member y of S for which $z <_A y$.

The terms *minimal element* and *minimum element*, and the terms *maximal element* and *maximum element*, are easy to confuse. The concepts are closely related, but they are not the same. When we say that x is minimal in X we are not saying that x is less than or equal every other member of X; we are only saying that X does not contain any members less than x. Similarly when we say that x is maximal in X we are saying that X does not contain any member greater than x.

Let's look at some examples.

Example 9.53. 1. Consider the partially ordered set $\mathcal{P}(\{1,2,3,4\})$ with the $\subseteq$ order. Let $X = \{\{1,2\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}$. In this set $\{1,2\}$ and $\{2,3,4\}$ are both minimal. There is no minimum element. The element $\{1,2,3,4\}$ is both a maximal element and a maximum element. The only lower bound for this set is $\emptyset$.

2. Consider the set $\mathbb{R}$ with the usual order. For the set $[1,3)$, 1 is both a minimal element and a minimum element. There are no maximal elements or maximum elements even though the set is bounded above.

3. Consider the set of integers under the divisibility order. In the subset P of all prime numbers, every member is both minimal and maximal, but there is no minimum or maximum.

Here are some important basic facts about minimal, minimum, maximal and maximum elements.

Proposition 9.54. *For any partially ordered set A and for any subset X of A we have:*

1. *A has at most one maximum element and at most one minimum element.*

2. *If A has a maximum element then that element is maximal in A, and there are no other maximal elements.*

3. *If A has a minimum element then that element is minimal in A, and there are no other minimal elements.*

Exercise 9.6.7. Prove Proposition 9.54.

Exercise 9.6.8. Prove or find a counterexample: For any partially ordered set A, subset S of A and $x \in S$, if x is the unique minimal element of S, then x is a minimum for S.

Remark 9.55. By the proposition, a subset X of a partially ordered set has at most one maximum and at most one minimum. If X has a maximum it is common to refer to it as "*the* maximum member of X" rather than "*a* maximum member of X". However, when talking about a maximal member of X we refer to it as "*a* maximal member of X" rather than "*the* maximal member of X", since there may be more than one of them.

To summarize, a subset X of A may have no minimal elements, exactly one minimal element, a finite number of minimal elements, or infinitely many minimal elements, and the situation is similar for maximal elements. However, X may have at most one minimum element and at most one maximum element. A minimum element must be minimal, and a maximum element must be maximal, but a minimal element is not necessarily a minimum and a maximal element is not necessarily a maximum.

In the special case that the partially ordered is a total ordered set, then the notions of minimal and minimum are the same, and the notions of maximum and maximal are the same:

Proposition 9.56. *For any totally ordered set A and for any subset X of A:*

1. *A member x of X is a minimal element of X if and only if it is the minimum element of X.*

2. *A member x of X is a maximal element of X if and only if it is the maximum element of X.*

Exercise 9.6.9. Prove Proposition 9.56

Exercise 9.6.10. Let *Div* denote the partial order on the set of positive integers with the divisibility order. Consider the subset of integers $\{n \in \mathbb{N} : 3 \leq n \leq 30\}$. What are the minimal elements and maximal elements of this set in the poset *Div*.

9.6.2 The dual of a partial order.

For a general relation R, we defined $\overleftarrow{R}$, to be the reverse relation which is the relation obtained by exchanging the source and target and reversing the order of every pair. For homogeneous relations, the source and target are the same, so $\overleftarrow{R}$ is obtained by simply flipping the order of every pair. If we apply this to partial orders, this inverts the order of elements.

Proposition 9.57. *For any partial order relation R on A, $\overleftarrow{R}$ is also a partial order.*

The partial order $\overleftarrow{R}$ is called the **dual partial order** of R.

Exercise 9.6.11. Prove Proposition 9.57.

Example 9.58.

1. The dual of the $\leq$ relation on $\mathbb{R}$ is $\geq$.
2. The dual of the subset relation on $\mathcal{P}(A)$ is the superset relation.
3. The dual of the divisor relation on $\mathbb{N}$ is the "is a multiple" relation.

Any statement about the dual of a partial order can be translated into a statement about the partial order itself. The following proposition provides this translation for several basic concepts.

Proposition 9.59. *Suppose R is a partial order on A and $X \subseteq A$ and $x \in A$.*

1. *x is a minimal element of A with respect to $\overleftarrow{R}$ if and only if x is a maximal element of A with respect to R.*
2. *x is a maximal element of A with respect to $\overleftarrow{R}$ if and only if x is a minimal element of A with respect to R.*
3. *x is a minimum element of A with respect to $\overleftarrow{R}$ if and only if x is a maximum element of A with respect to R.*
4. *x is a maximum element of A with respect to $\overleftarrow{R}$ if and only if x is a minimum element of A with respect to R.*
5. *x is a lower bound for A with respect to $\overleftarrow{R}$ if and only if x is an upper bound of A with respect to R.*
6. *x is an upper bound of A with respect to $\overleftarrow{R}$ if and only if x is a lower bound of A with respect to R.*

Exercise 9.6.12. Prove Proposition 9.59

9.6.3 Selecting a minimal or maximal element

In the proof of Theorem 7.5, the integer q is introduced into the proof using the instruction: "Let q be the smallest member of D". Introducing the smallest (or largest) member of a set of numbers into a proof is often a useful technique. More generally, if instead of having a subset of numbers, we have a subset from some partially ordered set it can be useful to select a minimal or maximal member of the set.

But this type of instruction is not always permissible. For example, if X is the set of real numbers and $A = (0, 1)$ then the instructions "Let m be the least member of A" and "Let m be a minimal member of A" are both invalid because A has no minimal member.

So it is important for us to identify conditions that allow us to select a minimal, minimum, maximal or maximum member of X. We start with:

Theorem 9.60. *Every non-empty finite partially ordered set has a minimal element and a maximal element.*

Proof. We will prove only that every finite poset has a minimal element; the proof for maximal elements is similar. We prove this using the principle of mathematical induction introduced in Chapter 7.3. To use induction we restate what we need to prove as: For all integers n, every poset with n members has a minimal element By induction we may assume that any poset with fewer than n members has a minimal element. We split into cases, $n = 1$ and $n > 1$.

Case 1. Assume $n = 1$. Then any poset of size 1 has a unique element which is necessarily minimal.

Case 2. Assume $n > 1$. Suppose X is a set of size n and P is a poset on X. We must prove that P has a minimal element. Let $x \in X$ be some element. If x is a minimal member of X we are done, so assume that x is not a minimal member of X. Let $Y = \{y \in X : y < x\}$. Since x is not minimal Y is non-empty. By definition Y is a subset of X that does not contain x, so $|Y| < n$. By the induction hypothesis the poset P restricted to Y has a minimal element. Let z be a minimal member of Y. We claim z is also a minimal member of X. Suppose for contradiction that there is a member, call it w of Z such that $w < z$. Then by the definition of Y, $w \in Y$, so z is not minimal in Y, which is a contradiction. We therefore conclude that z is a minimal member of X. □

The above is a great tool in proving theorems about finite partially ordered sets. But infinite partially ordered sets need not have minimal or maximal elements.

Definition 9.61.
- A partially ordered set P on set X is said to be **well-founded** provided that every subset of X has a minimal element (with respect to the order $\leq_P$).
- A partially ordered set P on set X is said to be **dual-well-founded** provided that every subset of X has a maximal element (with respect to the order $\leq_P$).

Observe that P is dual-well-founded is equivalent to saying that the dual poset $\overleftarrow{P}$ is well-founded.

Knowing that a poset P on X is well-founded can be extremely useful. It tells us that whenever we have a nonempty subset of X, we may assume that X has a minimal element, and use this in our proof. Similarly knowing that it is dual-well founded says that we may assume that any nonempty subset has a maximal element.

For the special case that the poset P is a total order, there is a special term for well-founded posets.

Definition 9.62.
- A partial order on set X is said to be a **well-ordering** provided that it is a total order and is well-founded.
- A partial order on set X is said to be a **dual well-ordering** provided that it is a total order and is dual-well-founded.

Usually when giving a new definition we provided some examples of things that fit the definition, and then some examples of things that don't fit the definition. Here we'll reverse the order.

Example 9.63. Some examples of posets that are not well-founded.

1. The set $\mathbb{Z}$ with the usual $\leq$ order is not well-founded or dual-well-founded
2. The set of rational numbers with the usual $\leq$ order in the interval $[0, 1]$ is not well-founded or dual-well-founded.
3. The set of closed intervals or real numbers that are contained in $[0, 1]$ ordered by inclusion. is not well-founded or dual-well-founded.

Exercise 9.6.13. For each of the examples in Example 9.63 explain why it is not well-founded or dual-well-founded.

The following theorem provides several examples of well-founded sets.

Theorem 9.64.

- *Any subset of $\mathbb{Z}$ with the standard order that is bounded below is well-founded. In particular, $\mathbb{N}$ is well-founded. (This is known as the* ***well-ordering principle for $\mathbb{Z}$****.*
- *The set of lists is well-founded under the prefix ordering.*
- *For any set A, the set $\mathcal{P}_{\mathit{fin}}(A)$ is well-founded under the containment order. It is dual-well-founded if and only if A is finite.*

All of these results follow as a consequence of the following. Let us say that a partially ordered set P on X is **lower finite** provided that for every $x \in X$, $\{y \in X : y \leq x\}$ is finite and is **upper finite** provided that for every $x \in X$, $\{y \in X : y \geq x\}$ is finite.

Exercise 9.6.14.

For each poset described in Theorem 9.64, prove that it is lower finite.

Theorem 9.65.

1. *Every lower finite partially ordered set is well-founded.*
2. *Every upper finite partially ordered set is dual-well-founded.*

Proof. We prove Part 1, the proof of Part 2 is left as an exercise.

Suppose A is an arbitrary partially ordered set. Assume A is lower finite. We must show that A is well-founded. Suppose X is an arbitrary nonempty subset of A; we must show that X has a minimal member..

Let b is an arbitrary member of X. If b is a minimal member of X then we're done, so assume that b is not a minimal member of X. Let Y be the set consisting of those members of X that are less than or b. Then Y is finite (since A is lower-finite). Then by Theorem 9.60, the restriction of the partial order to Y has a minimal element which we'll call w. We claim that w is a minimal member of X that is less than or equal to b. By definition of Y, we know $w \leq b$. We must show that w is a minimal member of X. (**Comment**: *We chose w to be a minimal member of Y, but this does not automatically imply that w is a minimal member of X.*) We prove that w is a minimal member of X by contradiction. Suppose for contradiction that w is not a minimal member of X. Then there is a member z of X such that $z < w$. Since $z \leq w$ and $w \leq b$, by transitivity we have $z \leq b$. Since $z \in X$ and $z \leq b$ then $z \in Y$. But then z is a member of Y that is less than w, and this contradicts that w is a minimal member of Y. So we conclude that w is a minimal member of X, as required to complete the proof of (1). □

Exercise 9.6.15. Prove Part 2 of Theorem 9.65.

An obstruction to well-foundedness. The property of well-foundness for a poset guarantees that *every subset* of the poset has a minimal element. The simplest example of a subset of a poset that does not have a minimal element is a **infinite descending chain**.

Definition 9.66.
- An **infinite descending chain** in a poset is an infinite sequence of elements $(x_i : x \in \mathbb{N})$ satisfying that for all $i \in \mathbb{N}$, $x_i > x_{i+1}$. Informally we write $x_1 > x_2 > x_3 > \dots$.
- An **infinite ascending chain** in a poset is an infinite sequence of elements $(x_i : x \in \mathbb{N})$ satisfying that for all $i \in \mathbb{N}$, $x_i < x_{i+1}$. Informally we write $x_1 < x_2 < x_3 < \dots$.

Proposition 9.67. *An infinite descending chain has no minimal element.*

Exercise 9.6.16. Prove Proposition 9.67

Example 9.68.
1. $(-i : i \in \mathbb{N})$ is an infinite descending chain in $\mathbb{Z}$ with the usual order.
2. $(1/2^i : i \in \mathbb{N})$ is an infinite descending chain in the interval $(0, 1)$ with the usual order
3. $([\frac{1}{2} - \frac{1}{i+1}, \frac{1}{2} + \frac{1}{i+1}]$ is an infinite descending in the poset of closed intervals ordered by containment.

By Proposition 9.67, a poset that contains an infinite descending chain has a subset that has no minimal element, and is therefore not well-founded. We therefore say that an infinite descending chain is an **obstruction** to being well-founded. The following theorem states that this is the only obstruction:

Theorem 9.69.
1. *Any poset with no infinite descending chain is well-founded.*
2. *Any poset with no infinite ascending chain is dual-well-founded.*

This theorem is a Corollary of a famous result in set theory which is discussed in the next section.

9.6.4 Zorn's Lemma (Optional)

The following theorem is extremely important and gets used extensively throughout mathematics. It is known commonly as Zorn's Lemma (after the mathematician Max Zorn who proved it in 1935, although Kazimierz Kuratowski proved it earlier.) We will use this lemma later to prove an important theorem in set theory, Theorem 13.19.

Recall that if P is a poset on X and $S \subseteq X$, a *lower bound for* S is a member of X that is less than or equal to every member of S and an *upper bound for* S is a member of X that is greater than or equal to every member of S.

Theorem 9.70. ***(Zorn's Lemma)*** *Suppose P is a partial order on X.*

1. *If every infinite descending chain of X has a lower bound in X then X has a minimal element.*

2. *If every infinite ascending chain of X has an upper bound in X then X has a maximal element.*

The proof of this theorem is beyond the scope of this course. It is commonly covered in a course of mathematical logic.

Let us use this theorem to prove Theorem 9.69.

Proof of Theorem 9.69. We prove Part 1 and leave the second part as an exercise.

Proof of Part 1. Suppose P is a poset on X with no infinite descending chain. We must show that P is well-founded which means that every subset of A of X has a minimal element. Suppose A is an arbitrary subset of X. Let $P[A]$ be the poset P restricted to the set A. We must show that $P[A]$ has a minimal element. Since P has no infinite descending chain and $A \subseteq X$. we have that $P[A]$ has no infinite descending chain. Since $P[A]$ has no infinite descending chain, it vacuously satisfies the requirement that every infinite descending chain has a lower bound. Therefore by the first part of Theorem 9.70, $P[A]$ has a minimal element. □

Exercise 9.6.17. Prove the Part 2 of Theorem 9.69.