MATH 642:587–Fall, 2011
Assignment 1(Version: October 5, 2011)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/∼saks/homework.html

2. Most of these problems are based on results that have appeared in research papers. Problems typically are broken into multiple steps that guide you through the solution.

3. For some problems hints are provided at the end of the assignment. This is noted by the statement "Hint given below." I recommend you try to solve the problem without the hint and refer to the hint if you are stuck. If you use a hint you should acknowledge using it in your acknwowledgements for the problem (as described in the general instructions for homework mentioned above.

4. Please be on the look out for errors in the problem statements. If something seems not to make sense, check with me before investing a lot of time on the problem. I would appreciate being notified of any typos (even minor ones).

PROBLEMS

1. Prove that for every boolean function f there is a layered branching program of width 3 that computes f.

2. The purpose of this problem is to prove the following **Theorem** (L. Valiant): There is a constant c such that for every odd integer n, there is a monotone boolean formula that computes MAJ_n, whose depth is at most $c \log n$.

   (a) Consider the set of (fan-in 3) circuits of the following form. There are n input nodes. The non-input nodes are arranged in the form of a balanced rooted ternary tree of some depth h (so there are 3^h leaves), with all edges directed towards the root. Each input node has an unrestricted number of edges directed into leaves of the tree, but each leaf has exactly three edges coming in. The function associated to each non-input node is MAJ_3. Prove that for some $h = O(\log n)$, there is a circuit of this form that computes MAJ_n. (Hint provided below)

   (b) Complete the proof of the Theorem.

3. The purpose of this problem is to exhibit a family of boolean functions f for which $bs(f) = \Theta((s(f))^2)$ square of its sensitivity. For $1 \leq k \leq n-2$, let $c_{n,k}(x_1, \ldots, x_n)$ denote the function that is 1 if there is an index $j \in [n-k-1]$ so that $x_i = 1$ for $i \in [j+1, j+k]$ and $x_j = x_{j+k+1} = 0$.

   (a) Prove that $s_1(c_{n,k}) \leq k+2$ and $s_0(c_{n,k}) = O(n/k)$.

   (b) Show that there is a function on $\Omega(n)$ variables obtained from $c_{n,k}$ by substitution which is sensitive to every variable at the all 1 input.

4. In this problem, you will prove the following theorem of Simon: If f is an n-variate boolean function that depends on all of its variables, then $s(f) = \frac{1}{2}(log_2 n - \log_2 \log_2(n))$

   Define H_m to be the *Hamming graph* which has vertex $\{0,1\}^m$ and edges between pairs of vertices that differ on exactly one coordinate. We say an edge xy is in direction i if x and y differ on coordinate i.

   (a) Prove the following **Lemma**: If $W \subseteq \{0,1\}^m$ is nonempty and G is the subgraph of H_n induced on W, and every vertex in G has degree at least d, then $|W| \geq 2^d$.

   (b) Let W_i be the set of vertices x such that $x_i = 0$ and $f(x) \neq f(x \oplus e^i)$. Prove that $|W_i| \geq 2^{n-2s(f)+1}$.

   (c) Prove the theorem.

5. A supersink of a directed graph G is a vertex v that has no edges coming out and has an edge coming in from every other vertex. Show that there is a decision tree for testing whether a digraph on $[n]$ has a supersink that uses at most $O(n)$ questions.

6. Prove that for any monotone boolean function f, $s_0(f) = bs_0(f) = c_0(f)$ and $s_1(f) = bs_1(f) = c_1(f)$.

7. The purpose of this problem is to exhibit an infinite sequence boolean functions f for which $deg(f) \leq s(f)^c$ for some constant $c < 1$.

   (a) Determine the degree and sensitivity of the function $E(x_1, x_2, x_3)$ that if 1 if all three variables are equal.

   (b) For $j \geq 1$, E^j be the function on 3^j variables which is defined by $E^1 = E$ and for $j \geq 2$, $E^j(x) = E(E^{j-1}(x^1), E^{j-1}(x^2), E^{j-1}(x^3))$ where x^1, x^2, x^3 are obtained by splitting the input into three substrings of equal size. Determine the degree and sensitivity of E^j.

8. Prove the following theorem of Rivest and Vuillemin (1976). Suppose $n = p^k$ for some prime p and integer k. Suppose that f is a boolean function on n variables, f is weakly symmetric and that $f(\underline{0}) \neq f(\underline{1})$. Then f is evasive(!) (Hint provided below)

Hints

Problem 2 Consider a "random" circuit C of this form where each input node is independently assigned one of the n variables uniformly at random. Consider, for each $a \in \{0,1\}^n$ the probability that C incorrectly computes the majority function on a.

Problem 4 For part (a): Proceed by induction on n

For part (b): Define an appropriate graph on W_i and show that every vertex has degree at least $n - s(f) + 1$, then apply part (a).

Problem 8 Let G be the symmetry group of f and let Π be the partition of $\{0,1\}^n$ into orbits under the action of G. Prove that there are exactly two orbits whose size is not divisible by p. Then apply the parity balance criterion.)