MATH 642:582–Fall, 2008

Assignment 5–Due Wednesday December 17 noon (Version: December 15)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/ saks/homework.html

2. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem.

Problems to be handed in.

1. Fix a positive integer p. Find an exact expression for the number of permutations of $[n]$ that have no cycle of length exactly p. (Your expression will be a sum, but it should be as simple as possible). Determine the asymptotics of your expression as n tends to ∞.

2. Let $A_1, \dots, A_n$ be events in a probability space and for $J \subseteq [n]$, let $A_J = \cap_{j \in J} A_j$. Let $p_J = \mathbf{Prob}[A_J]$.

   (a) Determine (with proof) a function $c(n, t, j)$ (in as simple form as possible) so that the probability that exactly t events occur is equal to $\sum_{J \subseteq [n]} c(n, t, |J|) p_J$.

   (b) Let T denote the random variable which is the number of A_j that occur. Prove that for any real number α, $\mathbf{E}[\alpha^T] = \sum_{J \subseteq [n]} p_J (\alpha - 1)^{|J|}$.

   (c) Give a formula for the number of permutations of $[n]$ with exactly t fixed points, and determine the asymptotics (for fixed t) as n tends to ∞. (This uses part (a) but not (b)).

3. In class it was shown that the edge-reconstruction conjecture for graphs is true for all graphs G with $|E(G)| > \frac{1}{2}\binom{|V(G)|}{2}$. In this problem, this result will be extended to graphs for which $|E(G)| > 1 + \log_2(|V(G)|!)$.

   More precisely, let n, m be fixed and consider graphs on vertex set $V = [n]$ having m edges. Let G_1, G_2 be two graphs having m edges and let $E(G_1) = \{e_1, \dots, e_m\}$ and $E(G_2) = \{f_1, \dots, f_m\}$. For $i \in \{1, 2\}$ and $j \in [m]$, let $G_i^j = G_i - \{e_j\}$. Assume that for all $j \in [m]$, G_1^j is isomorphic to G_2^j. The goal is to prove: if $2^{m-1} > n!$ then G_1 is isomorpnic to G_2.

   Suppose σ is chosen uniformly at random from the permutations of $[n]$. Let A_i (resp. B_i) be the event that σ maps the endpoints of edge e_i (resp. f_i) to vertices that are not adjacent in G_1. (Note: the lack of symmetry in these definitions with respect to G_1 and G_2 is intentional.)

   For $J \subseteq [m]$, let $A_J = \cap_{i \in J} A_i$ and $B_J = \cap_{i \in J} B_i$. For $0 \leq j \leq m$, let $a_j = \sum_{|J|=j} \mathbf{Prob}[A_J]$ and let $b_j = \sum_{|J|=j} \mathbf{Prob}[B_J]$

   Let Y denote the random variable that counts the number of $i \in [m]$ such that A_i occurs and Z denote the number of $i \in [m]$ such that B_i occurs.

(a) Show that if $a_j = b_j$ for all $j \in \{0, 1, \ldots, m\}$ then G_1 is isomorphic to G_2.

(b) Prove that if G_1^k is isomorphic to G_2^k for all $k \in [m]$ then for all $j \leq m-1$, $a_j = b_j$.

(c) Prove that $(b_m - a_m) = (-1/2)^m \mathbf{E}[(-1)^Y - (-1)^Z]$.

(d) Prove that if $2^{m-1} \geq n!$ then $b_m = a_m$.

(e) Put the parts together to prove the theorem.

Hints are given below.

4. (a) Consider a probability space Ω, and subsets (events) $A_1, A_2, \ldots, A_n$. As usual, for $I \subseteq [n]$, define $A_I = \cap_{i \in I} A_i$ and let $p_I = \mathbf{Prob}[A_I]$. Let $q_I = \mathbf{Prob}[A_I - \bigcup_{i \in [n]-[I]} A_i]$. For each I, express q_I in terms of $\{p_J : J \subseteq [n]\}$.

   (b) Consider a finite probability space Ω, and subsets (events) $A_1, A_2, \ldots, A_n$. As usual, for $I \subseteq [n]$, define $A_I = \cap_{i \in I} A_i$ and let $p_I = \mathbf{Prob}[A_I]$. Let $B_1, B_2, \ldots, B_n$ be another set of events and define $q_I = \mathbf{Prob}[B_I]$. Suppose that $p_I = q_I$ for all $I \neq [n]$ but $p_{[n]} \neq q_{[n]}$. Prove that Ω must have at least 2^{n-1} elements. (Hint: Part (a) may be helpful.)

   (c) Give an example to show that the bound in the first part is best possible.

5. A *Hamiltonian path* in a graph $G = (V, E)$ is a permutation of the vertices $v_1, \ldots, v_n$ such that each pair v_i, v_{i+1} is adjacent in G for $1 \leq i < n$. A naive algorithm for testing whether a graph has a Hamiltonian path requires checking all possible permutations of the vertex set, and thus requires $n!poly(n)$ number of steps, where $poly(n)$ denotes an unspecified polynomial function of n. The purpose of this problem is to derive an algorithm that runs in $2^n poly(n)$ steps (and also requires memory size only $poly(n)$). (Without being too precise here about what a "computation step" is, we assume that adding, subtracting or multiplying two k bit numbers can be done in $poly(k)$ computation steps.)

   (a) A walk in a graph is a sequence $v_1, \ldots, v_k$ of vertices, possibly with repetition such that each pair v_i, v_{i+1} is adjacent in the graph. For vertices x, y and integer k let $W_G(x, y, k)$ denote the number of k step walks from x to y. Give a method for computing $W_G(x, y, k)$ that uses at most $poly(n, k)$ steps.

   (b) Let $H_G(x, y)$ denote the number of Hamiltonian paths in G. Use the answer to the previous problem and inclusion-exclusion to derive a formula for computing $H_G(x, y)$ that uses at most $2^n poly(n)$ computation steps.

6. In this problem, we'll show that the chromatic number of an n-vertex graph can be computed in time $2^n poly(n)$.

   Let G be a given graph on vertex set V, with $|V| = n$.

   (a) Let $f : 2^V \longrightarrow \{0, 1\}$ be the function with $f(W) = 1$ if and only if W is independent. Describe how to construct a function table for f in time $poly(n)2^n$.

   (b) For $0 \leq i \leq n$, define functions $f_i : 2^V \longrightarrow \mathbb{N}$ by: $f_0 = f$ and for $1 \leq i \leq n$, and for $W \subseteq V$, $f_i(W) = f_{i-1}(W)$ if $i \notin W$ and $f_i(W) = f_{i-1}(W) + f_{i-1}(W - \{i\})$ if $i \in W$. Let $g = f_n$. Prove that $g(W)$ is equal to the number of subsets of W that are independent sets of G. Also, show that a table for g can be constructed in time $poly(n)2^n$.

(c) Let $c_k(G)$ denote the number of sequences $I_1, \ldots, I_k$ of k independent sets whose union is V. Express $c_k(G)$ in terms of the function values of the function g defined in the previous part.

(d) Explain how to compute the chromatic number of G using at most $poly(n)2^n$ arithmetic operations.

Some hints

Problem 3 For part (a): Use inclusion exclusion to express the probability that σ is an isomorphism from G_1 to itself, and to express the probability that σ is an isomorphism from G_1 to G_2.

For part (c): use the second part of problem 2.

Problem 6 For part (c), for each vertex v, let A_v be the k-tuples $I_1, \ldots, I_k$ for which v belongs to none of the sets I_j and use inclusion-exclusion.