MATH 642:582–Fall, 2008
Assignment 3–Due November 7 (Version: November 3)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/ saks/homework.html

2. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem.

3. This homework uses the following definitions. Let P be a partially ordered set.

   - The comparability graph of P is the undirected graph with vertex set P and xy an edge if and only if $x < y$ or $y < x$. We say that P *is connected* if and only if its comparability graph is connected.
   - For $x, y \in P$, the *interval* $[x, y]$ is the set of elements $\{z \in P : x \leq z \leq y\}$. In particular $[x, y]$ is empty if x is not less than or equal to y. We say that y *covers* x if $[x, y] = \{x, y\}$.
   - The cover digraph of P (also called the Hasse diagram) is the directed graph with arc set $\{(y, x) : y \text{ covers } x\}$.
   - A rank function for P is a function r from P to $\mathbb{Z}$ with the property that whenever y covers x, $r(y) = r(x) + 1$. We call the pair (P, r) a *ranked poset* or *graded poset.* If P has a rank function we often say P *is a ranked poset*, leaving the rank function implicit.
   - If r is a rank function of P, and $j \in \mathbb{Z}$ the *jth level of* P with respect to r is $r^{-1}(i)$.
   - If (P, r) is a finite ranked poset then the sequence $(w_i : i \in \mathbb{Z})$ where $w_i = w_i(P, r) = |r^{-1}(i)|$ is called the rank sequence of (P, r) (or sometimes the *Whitney numbers of the second kind.*)
   - If we say P is a ranked poset with level sequence $P_s, P_{s+1}, \ldots, P_t$ we mean that the rank function associated to P is the function sending P_i to i.
   - A ranked poset (P, r) is said to be *rank unimodal* if its rank sequence $(w_i :\in \mathbb{Z})$ is unimodal, that is for some s we have $w_{i-1} \leq w_i$ for $i \leq s$ and $w_{i-1} \geq w_i$ for $i > s$. (P, r) is *rank symmetric* if for some k we have $w_i = w_{k-i}$ for all i.
   - P satisfies the Jordan-Dedekind chain condition (JDC) if for any pair of elements $x \leq y$, every maximal chain from x to y has the same size.
   - If P is a ranked poset with level sets $P_0, \ldots, P_k$ a chain C of P is a *rank symmetric chain* if for some $i \leq k/2$, P contains one element from each of the levels $P_i, P_{i+1}, \ldots, P_{k-(i+1)}, P_{k-i}$. A *symmetric chain decomposition* of P is a partition of P into symmetric chains. P is a *symmetric chain order* if it has a symmetric chain decomposition.
   - For a poset P, a Sperner k-family is a subset of elements that contains no chain of size $k+1$; equivalently it is a subset that can be covered by k antichains. $a_k(P)$ denotes the size of the largest Sperner k-family.

- For a ranked poset P with level sequence $P_0, \ldots, P_s$, we say that P has the *Sperner property* if $a_1(P) = \max_i |P_i|$, and has the *k-Sperner property* if $a_k(P)$ is equal to the maximum of $|P_{i_1}| + \cdots + |P_{i_k}|$ over all sequences $0 \leq i_1 < i_2 < \cdots < i_k \leq s$. P has the *Strong Sperner property* if it is k-Sperner for all k.
- The product of posets P, Q, $P \times Q$ is the poset with elements (x, y) with $x \in P$ and $y \in Q$ and order $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

Some warm-up exercises NOT TO BE HANDED IN. You may refer to these results in solving later problems)

1. If P is finite, prove that $x < y$ if and only if there is a directed path from x to y in the cover graph.
2. Show that the previous result need not hold if P is infinite.
3. Prove that if P is a finite ranked connected poset then for each integer a, there is a unique rank function satisfying $a = \min_{x \in P} r(x)$. (Also show that this is not true if P is disconnected.)
4. Prove that if P is finite and has the property that every maximal chain has the same size, then P satisfies the JDC condition.
5. Prove that if P is finite and satisfies the Jordan Dedekind chain condition then P is ranked.
6. If (P, r) and (Q, s) are ranked posets then the function t on $P \times Q$ given by $t(x, y) = r(x) + s(y)$ is a rank function on $P \times Q$.

Problems to be handed in.

1. In the following, P denotes a ranked poset with rank function r and nonempty level sets $P_0, \ldots, P_k$.

   - A bipartite graph $G = (V, W; E)$ is said to have the *normalized matching property* if for all $X \subseteq V$, $|N(X)|/|W| \geq |X|/|V|$. P is said to have the *normalized matching property* if the bipartite graph G_i between each pair of successive levels P_i and P_{i+1} has the normalized matching property, for each i.
   - A family of chains $\mathcal{C}$ (repeated chains allowed) is said to be a *uniform chain cover* of P if (i) each chain of $\mathcal{C}$ contains one element from each level and (ii) for each $p, q \in P$ such that $r(p) = r(q)$, p and q belong to the same number of chains of $\mathcal{C}$.
   - P has the LYM property if for any antichain A, $\sum_{x \in A} \frac{1}{|P_{r(x)}|} \leq 1$.

   (NOT TO HAND IN: Prove that if P is a finite connected ranked poset satisfying the LYM property then P has the Strong Sperner property.)

   Prove that the following three conditions on a ranked poset P are equivalent:

   (a) P has the normalized matching property

(b) P has a uniform chain cover

(c) P has the LYM property.

(Hint on last page.)

2. Consider the unit hypercube C_n in n dimensions which is the convex hull of the set of points $\{0,1\}^n$, which are the *vertices* of C_n. Two vertices x, y are *adjacent* in C_n if they differ in exactly one coordinate. An edge of C_n is a line segment joining two adjacent points; for adjacent x, y the edge joining x and y is denoted $e(x, y)$. If $a \in R^n$ and $b \in R$, the hyperplane $H_{a,b}$ is the set of points $x \in R^n$ satisfying $a \cdot x = b$. $H_{a,b}$ is said to *slice* the edge $e(x, y)$ if x and y are on opposite sides of the hyperplane, i.e., $a \cdot x - b$ and $a \cdot y - b$ are nonzero and have opposite sign.

   (a) Determine the maximum number of edges that can be sliced by a single hyperplane. (Hint on last page.)

   (b) Let $f(n)$ be the minimum number of hyperplanes needed to slice every edge. Prove that $f(n) \le n$ and that $f(n) = \Omega(\sqrt{n})$. (Note: The lower bound on $f(n) = \Omega(\sqrt{n})$ is the best currently known. The upper bound has been improved slightly to $f(n) \le \lceil 5n/6 \rceil$. It is a very nice open problem to improve either of these bounds.)

3. Let n, k be positive integers with $n \ge 2k$. Suppose $\mathcal{A}$ is an antichain of $2^{[n]}$ of size at least $\binom{n}{k}$. Prove that the average size of a member of $\mathcal{A}$ is at least k. (Hint on last page.)

4. (a) Prove that if P is a ranked poset with level sets $P_0, \ldots, P_k$ and P has a symmetric chain decomposition then P is rank unimodal, rank symmetric, and has the Strong Sperner property.

   (b) Prove that the product of two symmetric chain orders is a symmetric chain order.

   (c) Prove that the poset Div_n of divisors of n, ordered by divisibility, is a symmetric chain order.

5. Given a subset S of $\mathbb{R}^d$, a *linear dichotomy* is a partition (S_1, S_2) of S with the property that there is a pair (a, b) with $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that $\{x \in S : a \cdot x \le b\} = S_1$. Prove: For $S \subseteq \mathbb{R}^d$ of size n, the number of linear dichotomies is at most $\sum_{j=0}^{d+1} \binom{n}{j}$. (Hint on last page.)

6. For a hypergraph $\mathcal{H}$ on V, and $S \subseteq V$, the *neighborhood of S in $\mathcal{H}$*, denoted $(\mathcal{H} : S)$ is the hypergraph on $V - S$ with $E \in (\mathcal{H} : S)$ if and only if $E \cup S \in \mathcal{H}$. (In particular, the neighborhood of $\emptyset$ in $\mathcal{H}$ is just $\mathcal{H}$ itself.) Suppose that $\mathcal{H}$ is an r-uniform hypergraph, and that for every $S \subseteq V$, $\tau(\mathcal{H} : S) \le w$. Prove $\mathcal{H}$ has at most w^r edges.

7. (a) Recall that a sequence of disjoint set pairs (SDSP) is a sequence $(A_1, B_1), \ldots, (A_t, B_t)$ such that $A_i \cap B_i = \emptyset$. An SDSP is weakly crossing if for each $i \neq j$ $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$.

   (b) Prove that for a weakly crossing SDSP, $\sum_{i=1}^{t} 2^{-|A_i|+|B_i|} \le 1$.

   (c) Some preliminaries:

      - A set of points in $\mathbb{R}^d$ is in *convex position* if no one of the points is in the convex hull of the others.

- A j-simplex is the convex hull of $j+1$ distinct points in convex position.
- Warmup problem (not to be handed in): A d-simplex in $\mathbb{R}^d$ is closed and compact. The boundary of a d-simplex is the union of $d+1$ $(d-1)$-simplices (called the *facets* of the simplex) having disjoint interiors. Each facet of the simplex lies in a unique $d-1$-dimensional hyperplane, which is called a *bounding hyperplane* of the simplex.
- Two d-simplexes are said to be *adjacent* if there is a unique $d-1$ dimensional hyperplane containing their intersection. (Another warm-up: this hyperplane must be a bounding hyperplane of both simplexes.)

Prove that a collection of d-simplexes in $\mathbb{R}^d$ that are pairwise adjacent has size at most 2^{d+1}.

8. A *closure space* consists of a pair (X, λ) where X is a set and $\lambda : 2^X \longrightarrow 2^X$ satisfies:

   - $A \subseteq \lambda(A)$, for all $A \subseteq X$.
   - $A \subseteq B$ implies $\lambda(A) \subseteq \lambda(B)$, for all $A, B \subseteq X$.
   - $\lambda(\lambda(A)) = \lambda(A)$.

   We say that λ is a *closure on* X. A set $C \in image(\lambda)$ is called a λ-closed subset. Define $cl(\lambda)$ to be the set of λ-closed sets.

   A hypergraph $\mathcal{H}$ on X is called an *alignment on* X if (i) $X \in \mathcal{H}$ and (ii) $\mathcal{H}$ is closed under arbitrary intersections: for subset $\mathcal{H}' \subseteq \mathcal{H}$, $\bigcap_{A \in \mathcal{H}'} A \in \mathcal{H}$.

   Let X be a fixed (possibly infinite) set. Prove that the map that associates each closure λ (on X) to $cl(\lambda)$ is a bijection between the set of closures on X to the set of alignments on X.

9. The König-Hall theorem says that $\nu(\mathcal{H} = \tau(\mathcal{H})$ if the hypergraph $\mathcal{H}$ is a bipartite graph. In this problem and the next we consider the analogous problem for the class of k-uniform k-partite hypergraphs.

   A k-uniform k-partite hypergraph (k-UP hypergraph) $\mathcal{H}$ is a hypergraph whose vertex set V is partitioned into sets $V_1, \ldots, V_k$ such that every edge contains exactly one element from each set.

   It turns out that there are k-UP hypergraphs $\mathcal{H}$ such that $\nu(\mathcal{H}) < \tau(\mathcal{H})$. (Warm-up problem, not to be handed in: Prove that $\tau(\mathcal{H})/\nu(\mathcal{H}) \leq k$ for any k-UP hypergraph $\mathcal{H}$.) In this problem we will show that if q is a prime power then there is a $(q+1)$-UP hypergraphs with $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) = q$. The example is based on an important combinatorial/geometric/algebraic structure called a *finite projective plane*. Abstractly, a finite projective plane is a hypergraph satisfying the following axioms: (1) any two edges intersect in exactly on vertex, (2) for any two vertices there is a unique edge containing them (3) for some r, every edge has size $r+1$ and every vertex has degree $r+1$. This structure is called a finite projective plane of order r.

   It is known that if q is a prime power then there is a finite projective plane of order q. This construction will be reviewed below. Then we'll construct the desired $(q+1)$-UP hypergraph. We use the fact that there is exists a finite field of order q, which we'll denote by $\mathbb{F}$.

(a) Consider the vector space $\mathbb{F}^3$, and let V be the set of one-dimensional subspaces, and U be the set of two-dimensional subspaces. Prove that $|V| = |U| = q^2 + q + 1$.

(b) For each $u \in U$, let E_u be the set of one-dimensional subspaces contained in u. Prove that $\{E_u : u \in U\}$ is a projective plane of order q.

(c) Now suppose $\mathcal{P}$ is any finite projective plane of order q with vertex set W. Construct a hypergraph $\mathcal{H}$ as follows: fix $v \in W$ and let $V = W - \{v\}$. Let $\mathcal{H}$ be the set of edges of $\mathcal{P}$ that don't contain v. Prove that $\mathcal{H}$ is $(q+1)$-partite hypergraph, and that $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) = q$.

Remark: H. Ryser conjectured in the 1960's that for any k-UP hypergraph $\tau(\mathcal{H}) \leq (k-1)\nu(\mathcal{H})$. The case $k = 2$ follows from König's theorem, and Aharoni proved it for $k = 3$ in 1999. It remains open for all $k \geq 4$.

Some hints

Problem 1 Prove $(a) \longrightarrow (b) \longrightarrow (c) \longrightarrow (a)$. To prove $(a) \longrightarrow (b)$, consider first the case that all of the levels of P have the same size. Then generalize to the general case.

Problem 2 The first part can be reduced to proving that some poset has the Sperner property. Further hint: First consider the case that the vector a defining the hyperplane has all coordinates non-negative.

Problem 3

1. Argue that it is enough to consider the case $|\mathcal{H}| = \binom{n}{k}$.
2. Show that the optimal solution of the following LP is a lower bound on the average size of a member of $\mathcal{H}$: Variables $x_0, \ldots, x_n$, $\min 1/\binom{n}{k} \sum_i i x_i$ subject to the constraints $\sum_i x_i = \binom{n}{k}$, $\sum_i x_i / \binom{n}{i} \leq 1$, and $x_i \geq 0$ for all i.
3. Consider the dual LP.

Problem 5 It will be helpful to prove the following: Any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls have nonempty intersection. Further hint: Let $x_1, \ldots, x_{d+2}$ be the set of points, define $y_1, \ldots, y_{d+1}$ by $y_i = x_i - x_{d+2}$ and note that the y_i are linearly dependent. Yet another hint: To apply the previous hint, consider the theory of VC-dimension discussed in class.