MATH 642:582–Fall, 2008

Assignment 2–Due October 16 (Version: October 4)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/ saks/homework.html

2. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem.

3. "A Course in Combinatorics" Chapters 5 and 6 by van Lint and Wilson provides a general reference for the König-Hall theorem and Dilworth's theorem

4. This homework uses the following definitions:

   - If V is a set, a *hypergraph* $\mathcal{H}$ on V is a set of nonempty subsets of V. The members of $\mathcal{H}$ are called *edges* and V is called the *vertex set*.
   - If $\mathcal{H}$ is a hypergraph on V, then $\mathcal{H}^{\downarrow}$ is the hypergraph $\{W \subseteq V : \exists E \in \mathcal{H}, W \subseteq E\}$ and $\mathcal{H}^{\uparrow}$ is the hypergraph $\{W \subseteq V : \exists E \in \mathcal{H}, E \subseteq W\}$
   - A *matching* of $\mathcal{H}$ is a subset of edges that are pairwise disjoint. The *matching number* $\nu(\mathcal{H})$ is the size of the largest matching of $\mathcal{H}$.
   - A *fractional matching* is a nonnegative real-valued function w with domain $\mathcal{H}$ satisfying that for every vertex v, $\sum_{E:v \in E} w(E) \leq 1$. The *fractional matching number* $\nu^*(\mathcal{H})$ is the maximum of $\sum_{E \in \mathcal{H}} w(E)$ over all fractional matchings of $\mathcal{H}$.
   - A *vertex cover* (or *blocking set* of $\mathcal{H}$ is a subset C of vertices such that $C \cap E \neq \emptyset$ for all $E \in \mathcal{H}$. $\tau(\mathcal{H})$ is the minimum size of a vertex cover.
   - A *fractional vertex cover* of $\mathcal{H}$ is a nonnegative real valued function κ defined on V with the property that for every edge E, $\sum_{v \in E} w(v) \geq 1$. The *fractional cover number* $\tau^*(\mathcal{H})$ is the minimum of $\sum_v w(v)$ over all fractional vertex covers.
   - The incidence matrix $M = M(\mathcal{H})$ of $\mathcal{H}$ is the matrix with rows indexed by edges and columns indexed by vertices, where $M_{E,v} = 1$ if $v \in E$ and is 0 otherwise.
   - A hypergraph is *intersecting* if $E \cap F \neq \emptyset$ for all $E, F \in \mathcal{H}$ and is *k-wise intersecting* if $E_1 \cap \cdots \cap E_k \neq \emptyset$ for all $E_1, \ldots, E_k \in \mathcal{H}$.
   - For a graph $G = (V, E)$, a *clique* is a subset of vertices that are pairwise adjacent, and an *independent set* is a subset of vertices that are pairwise non-adjacent.

1. (a) For a hypergraph $\mathcal{H}$ be a hypergraph on V, let $\mathcal{H}^C$ denote the hypergraph consisting of all vertex covers of $\mathcal{H}$. Prove that $(\mathcal{H}^C)^C = \mathcal{H}^{\uparrow}$.

   (b) For a hypergraph $\mathcal{H}$,let $\mathcal{H}^A = \{W \subseteq V : \forall S \in \mathcal{H}, |W \cap S| \leq 1\}$. Let $G(\mathcal{H})$ be the graph on V whose edge set is the set $\{uv : \exists S \in \mathcal{H}, \{u, v\} \subseteq S\}$. Prove that $\mathcal{H}^A$ is equal to the hypergraph consisting of all independent sets of $G(\mathcal{H})$ and $(\mathcal{H}^A)^A$ is equal to the hypergraph of all cliques of $G(\mathcal{H})$.

2. A hypergraph $\mathcal{H}$ is *τ-critical* if for any edge E of $\mathcal{H}$, $\tau(\mathcal{H} - \{E\}) < \tau(\mathcal{H})$. Let $\mathcal{H}$ be a τ-critical hypergraph. Prove that either the edges of $\mathcal{H}$ are pairwise disjoint or $\tau^*(\mathcal{H}) < \tau(\mathcal{H})$.

3. Let $\mathcal{H}$ be a hypergraph on V and $M = M(\mathcal{H})$ be its incidence matrix (defined above).

   (a) Prove that if every square submatrix of M has determinant in $\{0, 1, -1\}$ then $\nu(\mathcal{H}) = \tau(\mathcal{H})$.

   (b) Prove that if $\mathcal{H}$ is a bipartite graph (so V can be partitioned into two sets V_1, V_2 and every edge contains one vertex from V_1 and one from V_2) then every square submatrix of M has determinant in $\{0, 1, -1\}$.

4. Dilworth's theorem says: for any partially ordered set P, the width of P (maximum size of an antichain) is equal to the minimum size of a cover of P by chains.

   For a finite set Γ, let $A[\Gamma] = \{A_\alpha : \alpha \in \Gamma\}$ an indexed collection of sets. For $J \subseteq \Gamma$ we write $A[J]$ for the subcollection $\{A_\alpha : \alpha \in J\}$. The defect of $A[J]$ is defined to be $\max\{0, |J| - |\cup_{\alpha \in J} A_\alpha|\}$. A selection function for $\mathcal{A}$ is a map f defined on Γ such that $f(\alpha) \in A_\alpha$. The *defect* of a selection function f is equal to $|\Gamma| - |Range(f)|$.

   The (defect form of) the König-Hall Marriage theorem says: for any finite indexed collection of sets $A[\Gamma]$, the minimum defect of a selection function is equal to the maximum deficiency of a subcollection. (Not to hand in: Show the equivalence between this theorem and König's theorem that $\nu(G) = \tau(G)$ for any bipartite graph G.)

   (a) Deduce the König-Hall Theorem from Dilworth's theorem.

   (b) Deduce Dilworth's theorem from the König-Hall Theorem.

5. (a) Suppose that $r \leq n$ and M is an $r \times n$ matrix satisfying:

   (i) all entries are in $[n]$.

   (ii) within any row or column the entries are distinct.

   Prove that there is an $n \times n$ matrix M' satisfying (i) and (ii) such that M is a submatrix of M'.

   (b) More generally, for $r, s \leq n$ show that if M is an $r \times s$ matrix satisfying (i) and (ii) then there exists an $n \times n$ M' satisfying (i) and (ii) such that M is a submatrix of M' if and only if if each integer between 1 and n appears at least $r + s - n$ times in M.

6. Let $A[\Gamma]$ be an finite indexed family of subsets of the set X. A $\Gamma \times X$ matrix M is *compatible* with $A[\Gamma]$ if for all $x \in X$ and $\alpha \in \Gamma$, if $x \notin A[\alpha]$ then $M[\alpha, x] = 0$.

   (a) Prove that $A[\Gamma]$ has an injective selection function if and only if there is a nonnegative matrix compatible with $A[\Gamma]$ whose row sums are each at least 1, and whose column sums are each at most 1.

   (b) Let $A[\Gamma]$ be an finite indexed family of subsets of X. For $x \in X$, let $deg(x) = |\{\alpha : x \in A_\alpha\}|$. Prove: if for every $\alpha \in \Gamma$ and $x \in A_\alpha$ $|A_\alpha| \geq deg(x)$, then $A[\Gamma]$ has an injective selection function.

7. (a) Prove: if $\mathcal{H}$ is an r-uniform $r + 1$-wise intersecting hypergraph, then $\tau(\mathcal{H}) = 1$.

(b) More generally, prove:if $\mathcal{H}$ is an r-uniform k-wise intersecting then $\tau(H) \leq \frac{r-1}{k-1}+1$. (A hint appears at bottom of assignment.)

(c) For each $r, k \in \mathbf{N}$ such that $k-1$ divides $r-1$, give an example of an r-uniform k-intersecting hypergraph $\mathcal{H}(k,r)$ such that $\tau(\mathcal{H}) = \frac{r-1}{k-1}+1$.

8. Consider $2^{[n]}$ with the usual containment order. Let $\mathcal{A}$ be a maximal antichain of $2^{[n]}$. Prove that $\mathcal{A}$ can be partitioned into two sets $\mathcal{X}$ and $\mathcal{Y}$ such that $(\mathcal{X}^{\downarrow}, \mathcal{Y}^{\uparrow})$ is a partition of $2^{[n]}$. (A hint appears at bottom of assignment.)

9. The purpose of this problem is to prove a generalization of the Marriage theorem (which, in particular, gives a different proof of Hall's theorem). For an indexed family $A[\Gamma]$ of sets, the surplus is $|\cup_{\alpha\in\Gamma} A_\alpha| - |\Gamma|$. The global surplus of $A[\Gamma]$ is defined to be the minimum surplus of any indexed subfamily $A[\Gamma']$ where $\Gamma' \subseteq \Gamma$.

   If $A(\Gamma)$ and $B(\Gamma)$ are indexed families of sets (with the same index set Γ) we say that B is a *pruning* of A if $B_\alpha \subseteq A_\alpha$ for all $\alpha \in \Gamma$.

   Theorem. Suppose $A[\Gamma]$ has global surplus $s \geq 0$. Then $A[\Gamma]$ has a pruning $B[\Gamma]$ with global surplus s and all of its sets of size exactly $s+1$.

   (Not to hand in: (i) Why is this a generalization of Hall's theorem? (ii) In the case that $s=1$ the conclusion is equivalent to: $A[\Gamma]$ can be pruned to a graph (all edges of size 2) that has no cycles,i.e., is a forest.)

   (a) If $\Gamma' \subseteq \Gamma$ is nonempty and the surplus of $A[\Gamma']$ is equal to the global surplus of $A[\Gamma]$, we say that Γ' is *surplus critical* for $A[\Gamma]$. Prove that if Γ_1 and Γ_2 are surplus critical for $A[\Gamma]$ then $\Gamma_1 \cap \Gamma_2$ is either empty or is also surplus critical for $A[\Gamma]$.

   (b) $\alpha \in \Gamma$ is *unshrinkable* for $A[\Gamma]$ if replacing A_α by any a proper subset reduces the global surplus of $\mathcal{H}$. Prove that if α is unshrinkable for $A[\Gamma]$ then then there is a surplus critical set Γ' that contains α, such that $A_\beta \cap A_\alpha = \emptyset$ for all $\beta \in \Gamma'$ with $\beta \neq \alpha$.

   (c) Prove the theorem.

Hints:

Number 7 For problem 7, part b. Prove by induction on j the claim that for $j \leq k$, there is a set of j edges whose intersection has size at most $r - (\tau(\mathcal{H})-1)(j-1)$.

Number 8 Let $\mathcal{X}$ be a minimal subset of $\mathcal{A}$ satisfying $\mathcal{A}^{\downarrow} - \mathcal{A} \subseteq \mathcal{X}^{\downarrow}$.