MATH 642:582–Fall, 2008

Assignment 1–Due September 25 (Version: September 22)

- Please read the guidelines for homework available on the course web page:

  http://www.math.rutgers.edu/ saks/582F08/

- Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem.
- Stanley, Enumerative Combinatorics, Chapter 1, is a good reference.

1. Define a total ordering on $2^{[n]}$ by $S < T$ if $|S| < |T|$ or if $|S| = |T|$ and the largest element in $S \oplus T$ (the symmetric difference $(S - T) \cup (T - S)$) belongs to T. (Not to be handed in: show that this relation is indeed a total order on $\mathcal{P}([n])$). Let $A = \{a_1, a_2, \ldots, a_k\}$ be a k-subset of $[n]$ with $a_1 < a_2 < \ldots < a_k$. Express (with explanation) the number of sets $B \in \binom{[n]}{k}$ with $B < A$, as a sum of binomial coefficients.
2. Consider the following process: start with the trivial partition of $[n]$ into one part. Then apply the following step: take a part of the partition that has more than one element and break it into two parts. Repeat this step until the resulting partition consists of n singleton blocks. In how many ways can this be process be carried out?
3. Let Q be the set of all univariate polynomials $p(x)$, that map integers to integers. Show that a degree n polynomial p belongs to Q if and only if it can be written in the form $\sum_{i=0}^{n} a_i \binom{x}{i}$, where the a_i are integers. (In other words the polynomials $\binom{x}{i}$ form a basis for the $\mathbf{Z}$-module Q.)
4. A *partition of an integer* n (as opposed to a partition of a set) is defined to be a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers that sum to n. The numbers $\lambda_1, \ldots, \lambda_k$ are the *parts* of λ.

   Let m be a positive integer. If λ is a partition of some integer, define $u_m(\lambda)$ to be the number of integers that occur at least m times in λ. Let $v_m(\lambda)$ be the number of parts that are equal to m. Show that, for fixed m, the sum of $u_m(\lambda)$ is equal to the sum of $v_m(\lambda)$, where both sums range over all partitions of some fixed integer n.
5. Recall that B_n is the number of partitions of the set $[n]$. Let C_n be the number of partitions of $[n]$ such that any two consecutive integers are in different blocks. The purpose of this problem is to give two proofs of (*) $C_n = B_{n-1}$, for all $n \geq 1$.
   (a) Prove (*) by giving an explicit bijection between the sets counted (and carefully prove that it is a bijection).
   (b) Use recurrence equations to give an alternative proof of (*)

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6. Determine (with explanation) the connection coefficients for expressing the rising factorials in terms of falling factorials. In other words, for each $n \geq 0$, find coefficients $(a_{n,k} : 0 \leq k \leq n)$ so that $x^{\overline{n}} = \sum_{k=0}^{n} a_{n,k} x^{\underline{k}}$. (Connection coefficients will be discussed in class.)

7. Note: The combinatorial functions $S(n, k)$ and $s(n, k)$ will be defined in class (and are also defined in Stanley's book.)

   (a) Show that $S(n, k) \sim k^n/k!$, provided that $k < \frac{n}{\ln n}$. (Here we assume that k is a function of n, which may be the constant function).

   (b) For fixed k, determine the asymptotic behavior of $s(n, n-k)$ as $n \longrightarrow \infty$.

   (c) (Bonus) The asymptotic behavior found in the previous part can be valid even if k is allowed to grow with n, as long as it does not grow too fast. Determine how fast k can grow without invalidating the formula.

8. Let H be the set of all real intervals of the form $I = [a, b)$ where $a < b$ and a, b are integers. Write $\ell(I)$ for the length of I, which is $b - a$. Let $W(n)$ be the set of all sequences $I_1 = [a_1, b_1), \ldots, I_k = [a_k, b_k)$ of intervals (where k is not fixed), such that (1) each $I_j \in H$, (2) $a_1 = 0$, (3) For each $j \in \{1, \ldots, k-1\}$, $I_j \cap I_{j+1} \neq \emptyset$ and (4) The sum of the lengths of the I_j is exactly n. Let $w(n) = |W(n)|$. (Thus $w(1) = 1$, $w(2) = 2$ and $w(3) = 6$.) The purpose of this problem is to determine the ordinary generating function for $(w(n))$ and also to find a simple recurrence relation for $w(n)$.

   (a) Let $W_r(n)$ be the set of sequences in $W(n)$ for which $b_1 = r$ and let $w_r(n) = |W_r(n)|$. Determine (with proof) constants $(c_i(r) : i \geq 1, r \geq 1)$ so that:

   $$w_r(n) = \sum_{i \geq 1} c_i(r) w_i(n - r),$$

   for all $1 \leq r < n$.

   (b) Digression: For a formal power series in two variables x, y, $H(x, y) = \sum_{n \geq 1} \sum_{r \geq 1} a(n, r) y^r x^n$ let L_1 be the operator mapping $H(x, y)$ to the univariate FPS $H(x, 1)$ and L_2 be the operator mapping $H(x, y)$ to $\frac{\partial H}{\partial y}|_{y=1}$. What conditions do we need to impose on H so that these operators make sense as formal power series operators?

   (c) Let $f(x, y) = \sum_{n \geq 1} \sum_{r \geq 1} w_r(n) y^r x^n$ and let $g(x) = L_1(f)$ and $h(x) = L_2(f)$. Determine (with explanation) explicit rational functions $r_1(x, y)$, $r_2(x, y)$ and $r_3(x, y)$ such that $f(x, y)$ can be expressed in the form:

   $$f(x, y) = r_1(x, y) + r_2(x, y) g(x) + r_3(x, y) h(x).$$

   (This is somewhat tricky; hints are available upon request.)

   (d) Apply L_1 and L_2 to both sides of the previous equation, and solve for $g(x)$ as a rational function.

   (e) Find a linear recurrence equation satisfied by $w(n)$.

   (f) (Open, I think.) Find a combinatorial explanation for the recurrence equation found in the previous part.