MATH 642:582–Fall, 2003

Assignment 5–Due November 25 (Version: November 18, 2003)

1. (a) Prove that if $\mathcal{H}$ is intersecting then $\mathcal{H}$ is 3-colorable.

   (b) Recall the definition of global surplus from the previous homework. Prove that if $\mathcal{H}$ has global surplus at least 1, then $\mathcal{H}$ is 2-colorable.

2. Jukna, Problem 7.7.

3. Jukna, Problem 7.10.

4. Jukna, Problem 7.13.

5. By a *boolean function* we mean a function $f : \{0,1\}^n \longrightarrow \{0,1\}$. A *boolean circuit* is an acyclic directed graph with the following additional attributes: (1) The graph has one sink node (node of outdegree 0) called the *output* (2) The graph has one or more source nodes (nodes of indegree 0) which are called *input nodes*. The input nodes are labeled by distinct boolean variables $x_1, x_2, \ldots, x_s$. (3) Every edge e is labeled by a distinct boolean variable z_e. (4) Every non-input node v is labeled by a boolean variable y_v and a boolean function f_v which depends only on the variables corresponding to the edges coming into it. A boolean circuit C computes a boolean function f_C in the variables $x_1, x_2, \ldots, x_s$ as follows. Given values for $x_1, x_2, \ldots, x_s$, assign values inductively to the edge and node variables z_e and y_v as follows: if e leaves an input node i, $z_e = x_i$. For each non-input node v, if all of the edges coming into it have been assigned values, y_v is given the value obtained by evaluating f_v, and then for each edge e leaving v, z_e is given value y_v. The value f_C is equal to y_w where w is the output node.

   The *depth* of a circuit is the maximum length of a directed path from some input node to the output node. A circuit is a *standard monotone* circuit if every non-input node has in-degree at most 2, and each of the functions associated to non-input nodes is either the boolean AND function or the boolean OR function. (Note there is no restriction on the out-degree). The *majority function* MAJ_n (for n odd) is the function on variables $x_1, \ldots, x_n$ which is equal to 1 if and only if more than half the variables are 1.

   The purpose of this problem is to prove the following result of L. Valiant: **Theorem**. There is a constant c such that for every n, there is a standard monotone circuit that computes MAJ_n, whose depth is at most $c \log n$.

   (a) Consider the set of (fan-in 3) circuits of the following form. There are n input nodes. The non-input nodes are arranged in the form of a balanced rooted ternary tree of some depth h (so there are 3^h leaves), with all edges directed towards the root. Each input node has an unrestricted number of edges directed into leaves of the tree, but each leaf has exactly three edges coming in. The function associated to each non-input node is MAJ_3. Prove that for some $h = O(\log n)$, there is a circuit of this form that computes MAJ_n. (Hint: consider a "random" circuit C of this form where each input node is independently assigned one of the n variables uniformly at random. Consider, for each $a \in \{0,1\}^n$ the probability that C incorrectly computes the majority function on a.)

   (b) Complete the proof of the Theorem.

6. Recall that a k-partite k–uniform hypergraph (or, more briefly, a k-partite k-graph) is a hypergraph whose vertex set is partitioned into k sets, such that each edge consists of one vertex from each set. Ryser's conjecture says: For $k \geq 2$, a k-partite k graph $\mathcal{H}$ satisfies $\tau(\mathcal{H}) \leq (k-1)\nu(\mathcal{H})$. The case $k = 2$ is equivalent to König's theorem for bipartite graphs. Aharoni recently proved this for the case $k = 3$. The purpose of this problem is to discover Aharoni's proof, which uses a hard result of Aharoni and Haxell (which is stated but not proved below).

   (a) For a hypergraph $\mathcal{H}$, and subset $\mathcal{K}$ of edges, the *$\mathcal{K}$-width* of $\mathcal{H}$ is defined to be the size of the minimum collection of edges of $\mathcal{H}$ whose union meets every edge in $\mathcal{K}$. The *matching-width* of $\mathcal{H}$, $mw(\mathcal{H})$ is the maximum $\mathcal{M}$-width of $\mathcal{H}$ over all matchings $\mathcal{M}$ of $\mathcal{H}$. Prove that for a hypergraph $\mathcal{H}$ with maximum edge size r, $\nu(\mathcal{H}) \leq r \times mw(\mathcal{H})$.

   (b) Now let I be a finite set and let $(\mathcal{H}_i : i \in I)$ be a collection of hypergraphs on a common vertex set. A *partial system of disjoint representatives (PSDJR)* of size k consists of a set $K \subseteq I$ of size k and a function f mapping each $i \in K$ to an edge of $\mathcal{H}_i$ in such a way that $f(i) \cap f(j) = \emptyset$ for all $i \neq j \in K$. If $I = K$, we say this a *system of disjoint representatives (SDJR)*. Aharoni and Haxell used non-trivial results in combinatorial topology to prove that the following condition is sufficient for the existence of an SDJR: For each subset $J \subseteq I$, the matching-width of $\bigcup_{j \in J} \mathcal{H}_j$ is at least $|J|$. (Extra credit: Find a simple proof of this!)

   Use this to prove that for each $d \geq 0$: If for each subset $J \subseteq I$, the matching-width of $\bigcup_{j \in J} \mathcal{H}_j$ is at least $|J| - d$, then $(\mathcal{H}_i : i \in K)$ has a PSDJR of size $|I| - d$.

   (c) Now, let $\mathcal{H}$ be a 3-partite 3-graph with vertex partition V_1, V_2, V_3. Construct from $\mathcal{H}$ a family $(\mathcal{H}_i : i \in V_1)$ of hypergraphs where $\mathcal{H}_i$ is the 2-partite 2-graph (bipartite graph) consisting of edges $\{v_2, v_3\}$ such that $\{i, v_2, v_3\} \in \mathcal{H}$. Reformulate Ryser's conjecture for $k = 3$ in terms of the collection $(\mathcal{H}_i : i \in I)$ of bipartite graphs.

   (d) Prove Ryser's conjecture for $k = 3$. (Hint: you need all three previous parts and also the fact that $\nu(G) = \tau(G)$ for all bipartite graphs).