MATH 642:582–Fall, 2003

Assignment 1–Due September 19 (Version: September 16)

1. Define a total ordering on $2^{[n]}$ by $S < T$ if $|S| < |T|$ or if $|S| = |T|$ and the largest element in $S \oplus T$ belongs to T. (Not for handing in: check that this relation is indeed a total order on $\mathcal{P}([n])$). Let $A = \{a_1, a_2, \ldots, a_k\}$ be a k-subset of $[n]$ with $a_1 < a_2 < \ldots < a_k$. Express (with explanation) the number of sets $B \in \binom{[n]}{k}$ with $B < A$, as a sum of binomial coefficients.

2. Consider the following process: start with the trivial partition of $[n]$ into one part. Then apply the following step: take a part of the partition that has more than one element and break it into two parts. Repeat this step until the resulting partition consists of n singleton blocks. In how many ways can this be process be carried out?

3. Recall that B_n is the number of partitions of a set of size n. Let C_n be the number of partitions of a set of size n such that any two consecutive integers are in different blocks. Prove that $C_n = B_{n-1}$, for all $n \geq 1$.

4. Let Q be the set of all univariate polynomials $p(x)$, that map integers to integers. Show that a degree n polynomial p belongs to Q if and only if it can be written in the form $\sum_{i=0}^{n} a_i \binom{x}{i}$, where the a_i are integers. (In other words the polynomials $\binom{x}{i}$ form a basis for the **Z**-module Q.)

5. Let m be a positive integer. If λ is a partition of the integer n, define $u_m(\lambda)$ to be the number of integers that occur at least m times in λ. Let $v_m(\lambda)$ be the number of parts that are equal to m. Show that, for fixed m, the sum of $u_m(\lambda)$ is equal to the sum of $v_m(\lambda)$, where both sums range over all partitions of some fixed integer n. (Note: a *partition of an integer* n is defined to be a nonincreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of integers that sum to n. The numbers $\lambda_1, \ldots, \lambda_k$ are the *parts* of λ.)

6. Determine (with explanation) the connection coefficients for expressing the rising factorials in terms of falling factorials. In other words, for each $n \geq 0$, find coefficients $(a_{n,k} : 0 \leq k \leq n)$ so that $x^{\overline{n}} = \sum_{k=0}^{n} a_{n,k} x^{\underline{k}}$. (Connection coefficients will be discussed in class.)

7. (a) Show that $S(n,k) \sim k^n/k!$, provided that $k < \frac{n}{\ln n}$. (Here we assume that k is a function of n, which may be the constant function).
   (b) For fixed k, determine the asymptotic behavior of $s(n, n-k)$ as $n \longrightarrow \infty$.
   (c) (Bonus) The asymptotic behavior found in the previous part can be valid even if k is allowed to grow with n, as long as it does not grow too fast. Determine how fast k can grow without invalidating the formula.

More on back . . .

8. Let k be a positive integer and let $c_1, \ldots, c_k$ be complex numbers. The *linear constant coefficient recurrence relation* defined by $c_1, \ldots, c_k$ in variables $\underline{z} = (z_n : n \in \mathbb{N})$ is the system consisting of the equations

$$z_n = \sum_{i=1}^{k} c_i z_{n-i},$$

for all $n \geq k$.

Prove the following:

Theorem. Let $c_1, \ldots, c_k$ be complex numbers with $c_k \neq 0$, and let $\underline{a} = (a_n : n \in \mathbb{N})$ be a sequence of complex numbers. Let $p(x)$ be the polynomial defined by: $p(x) = 1 - \sum_{i=1}^{k} x^i c_i$. Let $\prod_{i=1}^{j}(1 - \lambda_i x)^{m_i}$ be the factorization of p into linear factors (with $\lambda_1, \ldots, \lambda_j$ distinct). Then the following are equivalent:

(a) $\underline{a}$ is a solution to the recurrence relation defined by $c_1, \ldots, c_k$.

(b) The ordinary generating function of $\underline{a}$, $\sum_{n \geq 0} a_n x^n$ is equal to $q(x)/p(x)$ where $q(x)$ has degree at most $k - 1$.

(c) There are polynomials $w_1, w_2, \ldots, w_j$ where for $1 \leq i \leq j$, w_i has degree at most $m_i - 1$ such that for all $n \in \mathbb{N}$, $a_n = \sum_{i=1}^{j} w_i(n) \lambda_i^n$. (You may use, without proof, the following standard result from algebra: if $r_1(x), \ldots, r_h(x)$ are univariate polynomials that have no common root then there are polynomials $s_1(x), \ldots, s_h(x)$ such that $\sum_{i=1}^{h} s_1(x) r_1(x) = 1$.)