MATH 642:491–Fall, 2014

Suggested problems on inequalities

1. Prove that for any real number $x \geq -1$, and any positive integer n $(1+x)^n \geq 1+nx$.
2. Prove that $n! \geq (n/e)^n$ and that $n! \leq (n+1)\left(\frac{n+1}{e}\right)^n$.
3. Suppose that $a_1, a_2, \ldots$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that for any $p > 1/2$, $\sum_{n\geq\infty} \sqrt{a_n}/n^p$ also converges.
4. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and let s denote their sum. Show that $(1+a_1)(1+a_2)\ldots(1+a_n) \leq \sum_{i=0}^{n} s^i/i!$.
5. Let $p_1, \ldots, p_n$ be distinct points in the closed unit disc in the plane. Let d_k be the distance from p_k to the nearest other point. Show that $\sum_{k=1}^{n}(d_k)^2 \leq 16$.
6. For n positive real numbers with minimum m and maximum M, let A and G denote their arithmetic and geometric means. Prove that $A - G \leq (\sqrt{M} - \sqrt{m})^2/n$.
7. Let $x_1, \ldots, x_n$ be positive real numbers and k a positive integer. Prove $\frac{1}{n}\sum_i x_i^k \leq \frac{\sum_i x_i^{k+1}}{\sum_i x_i}$
8. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be nonnegative real numbers. Show $(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \leq [(a_1+b_1)\cdots(a_n+b_n)]^{1/n}$.
9. Let $x_1, \ldots, x_n$ be real numbers in $[0, \pi]$ Let x be their average. Prove that:
   $\prod_{i=1}^{n} \sin(x_i)/x_i \leq (\sin(x)/x)^n$.
10. If a, b, c are positive reals with $abc = 1$. Show that:
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$
11. If a, b, c are positive reals, show that:
$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$