640:477:03 – Solutions to Exam 2

1. Let X be a continuous random variable with probability density function $f(x) = C/x^3$ for $x \geq 1$ and $f(x) = 0$ otherwise.

   (a) Determine C.

   Solution. Since $f(x)$ is a density function: $\int_{-\infty}^{\infty} f(x)dx = 1$. By defintion of $f(x)$,

   $$\int_{-\infty}^{\infty} f(x)dx = \int_{1}^{\infty} C/x^3 dx = \frac{C}{-2}x^{-2}|_1^{\infty} = \frac{C}{2}$$

   Therefore $C = 2$.

   (b) Determine the cumulative distribution function for X.

   Solution. The cumulative distribution function $F(x)$ is 0 for $x < 1$ (since $f(x) = 0$ for $x < 1$) and for $x \geq 1$ is given by:

   $$\int_{-\infty}^{x} f(t)dt = \int_{1}^{x} \frac{2}{t^3}dt = \frac{-1}{t^2}|_1^x = 1 - \frac{1}{x^2}.$$

   (c) Determine the expected value of X.

   Solution.
   $E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\frac{2}{x^3}dx = -\frac{2}{x}|_0^{\infty} = 2.$

2. Let X be a random variable that is exponentially distributed with parameter 1. Suppose $Y = X^2$. Find the probability density function for Y.

   Solution. First we find the CDF of Y. To do this we need the CDF of X. The PDF of X is e^{-x} for $x \geq 0$ and 0 otherwise, so the CDF of X is 0 for $x < 0$ and for $x \geq 1$ is $F_X(x) = \int_0^x e^{-t}dt = -e^{-t}|_0^x = 1 - e^{-x}$.

   Now to get the CDF for Y we have that $F_Y(y) = 0$ for $y < 0$ and for $y \geq 0$:

   $F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt{y}]$.

   Since $P[X \leq 0] = 0$ we have that $P[-\sqrt{y} \leq X \leq \sqrt{y}] = P[0 \leq X \leq \sqrt{y}] = F_X(\sqrt{y}) = 1 - e^{-\sqrt{y}}$.

   Now the density function $f_Y(y)$ is the derivative of $F_Y(y)$ so is 0 for $y \leq 0$ and is $\frac{e^{-\sqrt{y}}}{2\sqrt{y}}$ for $y > 0$.

3. In the weekly state lottery a sequence of 3 random numbers from 1 to 100 are selected. When a player buys a lottery ticket, he chooses 3 numbers that are printed on the ticket. In a given week 2,000,000 tickets are sold. Assume that the numbers on each ticket were selected uniformly at random from 1 to 100.

   (a) Estimate the probability that there are 4 or more winning tickets.

   Solution. The probability of a particular ticket winning is $(1/100)^3 = 1/1,000,000$. The number of winning tickets is binomially distributed with parameters $n = 2,000,000$ and $p = 1/1,000,000$. Since n is large and $np = 2$ is small we can estimate the distribution of the number of winning tickets by the distribution of a Poisson RV Y with parameter 2. So our estimate is:

   $$\begin{aligned} P[Y \geq 4] &= 1 - P[Y = 0] - P[Y = 1] - P[Y = 2] - P[y = 3] \\ &= 1 - e^{-2} - 2e^{-2} - 4e^{-2}/2 - 8e^{-2}/6 = 1 - \frac{19}{3e^2}. \end{aligned}$$

(b) Estimate the conditional probability that there is exactly one winner given that there is at least one winner.

Solution. Using the approximation by the random variable Y as in the first part, we want

$$\frac{P[(Y \geq 1) \text{ AND}(Y = 1)]}{P[Y \geq 1]} = \frac{P[Y = 1]}{(1 - P[Y = 0])} = \frac{2e^{-1}}{1 - e^{-1}} = \frac{2}{1 + e}.$$

4. A piece of equipment is designed so that it is connected to 5 batteries, and it will function properly is at least 2 of the batteries are working. If the lifetime (in years) of each battery is exponentially distributed with parameter $1/2$, estimate the probability that the piece of equipment is still functioning properly after 3 years.

Solution. Let p be the probability that a particular battery survives for at least 3 years. Since the battery lifetime is exponential with parameter 1/2, $p = 1 - \int_0^3 \frac{1}{2} e^{-x/2} dx = 1 - e^{-x/2}|_0^3 = 1 - (1 - e^{-3/2}) = e^{-3/2}$.

Now the number of surviving batteries out of 5 batteries is binomially distributed with $n = 5$ and $p = e^{-3/2}$ so the probability of at least 2 batteries surviving is 1 - P(0 survive) - P(1 survives) which is:

$1 - (1 - e^{-3/2})^5 - 5(1 - e^{-3/2})^4 e^{-3/2}$.

5. A wheel of fortune has the numbers 1, 2 and 3 on it. Assume that for each spin of the wheel the probability that the outcome is 1 is 1/2, the probability that the outcome is 2 is 1/3 and the probability that the outcome is 3 is 1/6. Suppose we spin the wheel twice and let X be the minimum of the two numbers obtained and Y be the maximum.

(a) Find the joint probability mass function of X and Y.

Solution. The possible outcomes are (X, Y) are $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 2)$, $(2, 3)$ and $(3, 3)$. Let A_1, A_2 be the outcomes of spin 1 and 2.

$$\begin{aligned}
P[(X, Y) = (1, 1)] &= P[(A_1 = 1) \text{ AND}(A_2 = 1)] = (1/2)(1/2) = 1/4. \\
P[(X, Y) = (1, 2)] &= P[(A_1 = 1) \text{ AND}(A_2 = 2)] + P](A_1 = 2) \text{ AND}(A_2 = 1)] \\
&= (1/2)(1/3) + (1/3)(1/2) = 1/3. \\
P[(X, Y) = (1, 3)] &= P[(A_1 = 1) \text{ AND}(A_2 = 3)] + P](A_1 = 3) \text{ AND}(A_2 = 1)] \\
&= (1/2)(1/6) + (1/6)(1/2) = 1/6. \\
P[(X, Y) = (2, 2)] &= P[(A_1 = 2) \text{ AND}(A_2 = 2)] = (1/3)(1/3) = 1/9. \\
P[(X, Y) = (2, 3)] &= P[(A_1 = 1) \text{ AND}(A_2 = 3)] + P](A_1 = 3) \text{ AND}(A_2 = 1)] \\
&= (1/3)(1/6) + (1/6)(1/3) = 1/9. \\
P[(X, Y) = (3, 3)] &= P[(A_1 = 3) \text{ AND}(A_2 = 3)] = (1/6)(1/6) = 1/36.
\end{aligned}$$

(b) Determine the marginal mass function of X.

Solution.

$$\begin{aligned}
P[X = 1] &= P[(X, Y) = (1, 1)] + P[(X, Y) = (1, 2)] + P[(X, Y) = (1, 3)] = 3/4. \\
P[X = 1] &= P[(X, Y) = (2, 2)] + P[(X, Y) = (2, 3)] = 1/9 + 1/9 = 2/9. \\
P[X = 1] &= P[(X, Y) = (3, 3)] = 1/36.
\end{aligned}$$

6. Being extremely bored, we repeatedly play a game in which I select a card from a full deck of cards at random and you guess the suit of the card. If we play the game 200 times, estimate the probability that you are correct at least 60 times.

   Solution. Lert X be the number of times you are correct. X is binomially distributed with parameters $n = 200$ and $p = 1/4$. The probability that X is at least 60 is therefore $\sum_{j\geq 60} \binom{200}{j}(1/4)^j(3/4)^{200-j}$.

   This answer is not easy to compute (as specified in the instructions on the first page of the exam) so we look for an estimate. Since $np = 50$ is reasonably large and p is not close to 1, we can use a normal approximation. X has approximately the same distribution as the RV Y that is normally distributed with mean $np = 50$ and variance $np(1-p) = 37.5$. Using the continuity correction $P[X \geq 60]$ is approximately $P[Y \geq 59.5]$ Now $(Y - 50)/\sqrt{37.5}$ is a standard normal so $P[Y \geq 59.5] = 1 - \Phi((59.5 - 50)/\sqrt{37.5})$.

7. Suppose X and Y are jointly distributed random variables whose joint density function satisfies $f(x, y) = \frac{3}{8}(y^2 - x^2)$ where $0 \leq x \leq y \leq 2$ and 0 otherwise.

   (a) Sketch the region of the x-y plane for which the density is nonzero.

   Solution. The region is a triangle in the x, y plane with vertices (0,0), (0,2) and (2,2). (Sketch omitted.)

   (b) Find the marginal density function of X

   Solution $f_X(x) = \int_{y=-\infty}^{\infty} f(x, y)dy$. Now for a fixed value of x, $f(x, y)$ is 0 if $y < x$ or if $y > 2$ and is equal to $\frac{3}{8}(y^2 - x^2)$ for $x \leq y \leq 2$. Hence
   $f_X(x) = \int_{y=x}^{2} \frac{3}{8}(y^2 - x^2)dy = \frac{3}{8}(y^3/3 - x^2 y)|_{y=x}^{y=2} = 1 - \frac{3}{4}x^2 + \frac{1}{4}x^3$.

   (c) Determine the probability that $Y \geq 1$.

   Solution. The picture of part a is a good guide for setting up the integral. We want to integrate $f(x, y)$ over the trapezoid obtained by chopping off the part of the triangle below $y = 1$ so the outer integral integrates y form 1 to 2, and the inner integral integrates x from 0 to y:

$$\begin{aligned}
\int_{y=1}^{2} dy \int_{x=0}^{y} \frac{3}{8}(y^2 - x^2)dx &= \int_{y=1}^{2} dy \frac{3}{8}(y^2 x - x^3/3)|_{x=0}^{x=y} \\
&= \int_{y=1}^{2} dy \frac{y^3}{4} = \frac{y^4}{16}|_1^2 = \frac{15}{16}.
\end{aligned}$$

   (d) Find the expected value of $1/(x + y)$.

   Solution. We want to integerate over the triangle in part (a).

$$\begin{aligned}
\int_{y=0}^{2} dy \int_{x=0}^{y} dx \frac{1}{x+y}\frac{3}{8}(y^2 - x^2) &= \frac{3}{8}\int_{y=0}^{2} dy \int_{x=0}^{y} dx(y - x) \\
&= \frac{3}{8}\int_{y=0}^{2} dy(xy - x^2/2)|_{x=0}^{y} \\
&= \frac{3}{8}\int_{y=0}^{2} dy \frac{y^2}{2} \\
&= \frac{3}{8}\frac{y3}{6}|_0^2 = \frac{1}{2}.
\end{aligned}$$