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Exam 1 – Solutions[1]

1. Consider the following experiment. Five balls, numbered 1 to 5 are in a bag. A sequence of balls is selected without replacement. The number appearing on the first ball determines the total number of balls (including the first ball) that are selected from the bag.

   (a) (**6 points**) How many outcomes are there in the sample space.

   Solution. Separately count sequences according to the number they begin with. Sequences beginning with i are of length i so the rest of the sequence is a non-repeating sequence of length $i-1$ out of 4.

   - First ball 1: 1 sequence
   - First ball 2: 4 sequences
   - First ball 3: 4×3 sequences
   - First ball 4: $4 \times 3 \times 2$ sequences
   - First ball 5: $4 \times 3 \times 2 \times 1$ sequences

   The total is $1 + 4 + 12 + 24 + 24 = 65$.

   (b) (**6 points**) What is the probability that the ball numbered 3 is selected?

   Solution. Write E for the event that 3 is selected and for i between 1 and 5 write F_i for the event that i is selected first.

   We have: $P(E) = \sum_{i=1}^{5} P(EF_i) = \sum_{i=1}^{5} P(F_i)P(E|F_i)$.

   $P(F_i) = 1/5$ for each i.

   - $P(E|F_1) = 0$ since if the first ball is 1, no other balls are selected.
   - $P(E|F_2) = 1/4$ since if the first ball is 2, one more ball is selected from the remaining 4.
   - $P(E|F_3) = 1$, trivially.
   - $P(E|F_4) = 3/4$, since if the first ball is 4, 3 more balls are selected out of the remaining 4.
   - $P(E|F_5) = 1$ since if the first ball is 5, all the remaining balls are selected.

   So $P(E) = \frac{1}{5}(0 + 1/4 + 1 + 3/4 + 1) = \frac{3}{5}$.

2. Consider a group of 200 families in which 40 families have no children, 20 families have one child, 60 have two children, 60 of the families have three children, and 20 of the families have four children.

   (a) (**10 points**) If we pick a family uniformly at random, what is the expected number of children in the family?

   Solution. Let X be the number of children in the selected family. From the given data $P(X = 0) = 1/5$, $P(X = 1) = 1/10$, $P(X = 2) = 3/10$, $P(X = 3) = 3/10$ and $P(X = 4) = 1/10$. So

$$E[X] = \frac{1}{5}(0) + \frac{1}{10}1 + \frac{3}{10}(2) + \frac{3}{10}(3) + \frac{1}{10}(4) = 2.$$

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(b) (**5 points**) If we pick a child uniformly at random from all the children, what is the expected number of children in the selected child's family?

Solution. Let Y be the size of the family of the selected child.

- The number of children in families with 0 children is 0,
- The number of children in families with 1 child is 20
- The number of children in families with 2 children is 120,
- The number of children in families with 3 children is 180
- The number of children in families with 4 children is 80.
- The total number of children is $20 + 120 + 180 + 80 = 400$.

So $P(Y = 0) = 0$, $P(Y = 1) = 20/400 = 1/20$, $P(Y = 2) = 120/400 = 3/10$, $P(Y = 3) = 180/400 = 9/20$ and $P(Y = 4) = 80/400 = 1/5$. So,

$$E[Y] = \frac{1}{20}(1) + \frac{3}{10}(2) + \frac{9}{20}(3) + \frac{1}{5}(4) = 2\frac{4}{5}.$$

3. (Recall that every card in a deck of playing cards is labeled by a *rank* (Ace,King,Queen, Jack, or a number from 2 to 10) and a *suit* (spades,hearts,diamonds or clubs.)) If a random poker hand (consisting of 5 cards) is selected from a deck of cards, what is the probability that the hand is:

(a) (**7 points**) A flush? (A flush is a poker hand where all cards have the same suit.)

Solution. The sample space size is $\binom{52}{5}$. The number of possible flushes is 4 (choose the suit) time $\binom{13}{5}$ (choose 5 cards from the chosen suit). So the probability is $4 \times \binom{13}{5}/\binom{52}{5}$.

(b) (**8 points**) A 3-of-a-kind? (A 3-of-a-kind consists of three cards of the same rank together with two other cards whose ranks are different from each other, and are both different from the rank of the 3-of-a-kind.)

Solution. Again the sample space size is $\binom{52}{5}$. To choose a 3-of-a-kind, choose the rank of the 3-of-a-kind (13 choices) then choose 3 cards of that rank ($4 = \binom{4}{3}$ choices), then choose the ranks of the two remaining cards ($\binom{12}{2}$ choices) and the suits of those two cards (4×4 choices). This gives a probability of:

$$\frac{13 \times 4 \times \binom{12}{2} \times 16}{\binom{52}{5}}.$$

(c) (**5 points**) What is the probability that the hand contains at least one heart?

Solution. This equals $1-Prob(\text{no hearts})$. There are $\binom{39}{5}$ hands with no hearts so the probability is $1-\binom{39}{5}/\binom{52}{5}$.

(d) (**5 points**) Let X be the number of different suits represented in the hand. Carefully express X as a sum of Bernoulli random variables, and compute the expected value of X.

Solution. Let C be the bernoulli random variable which is 1 if the hand contains at least one club. Define D, H and S be the analogous random variables for diamonds, hearts and spaces. So $X = C + D + H + S$ and so $E[X] = E[C] + E[D] + E[H] + E[S]$. Since H is Bernoulli, $E[H] = prob[H = 1]$ which is the probability that there is at least one heart, and by the previous part this is $1-\binom{39}{5}/\binom{52}{5}$. Similarly $E[C] = E[D] = E[S] = 1-\binom{39}{5}/\binom{52}{5}$ so

$$E[X] = 4\left(1 - \frac{\binom{39}{5}}{\binom{52}{5}}\right).$$

4. A given coin is weighted so that the probability it flips to heads is r.

(a) (**8 points**) Give the probability mass function for the random variable which counts the total number of tails in 12 flips.

Solution. This is a binomial random variable with parameters 12 and $1-r$ so the PMF is: $p(k) = \binom{12}{k}(1-r)^k r^{12-k}$ for $k \in \{0, 1, \ldots, 12\}$.

(b) (**7 points**) Suppose we flip the coin until the first time heads comes up. Give the probability mass function for the random variable which counts the total number of tails.

Solution. For there to be exactly k tails the first k coins must be tails and the $k+1$st coin must be heads so the probability is $(1-r)^k r$ and the PMF is $f(k) = (1-r)^k r$ for all nonnegative integers k.

5. (**8 points**) There is a virus which has two closely related types, Type A and Type B. In the population, 5% are infected with type A, 15% are infected with type B and everyone else is infected with neither. (Assume no one is infected with both viruses.) We have a test for the virus. If the test is performed on someone infected with Type A, the test will always return postivie. If the test is performed on someone infected with Type B the test will return positive with probability $4/5$. If the test is performed on an uninfected person the test will return positive with probability $1/5$ If a person is selected at random and we perform the test, what is the conditional probability that the person is not infected (with either type A or type B) given that the test is positive.

Solution. Let E_A be the event that a randomly chosen person has virus A, E_B be the event that the person has virus B and E_N be the event that he/she has neither. Let Pos be the event that the test is positive. We are asked for $P(E_N|Pos)$. By Bayes' Theorem: $P(E_N|Pos) = P(Pos|E_N)P(E_N)/P(Pos)$.

We are given $P(Pos|E_N) = 1/5$ and $P(E_N) = 1 - (.05 + .15) = .8$. We have

$$\begin{aligned} P(Pos) &= P(E_A)P(Pos|E_A) + P(E_B)P(Pos|E_B) + P(E_N)P(Pos|E_N) \\ &= (.05)(1) + (.15)(4/5) + (.8)(1/5) = .33, \end{aligned}$$

so $P(E_N|Pos) = (1/5)(.8)/.33 = 16/33$.

6. (**7 points**) Suppose that A and B are events in a probability space and that the following information is given: $P(A) = .4$, $P(B) = .5$ and $P(A \cup B) = .7$. Find $P(B|A^c)$.

   Solution. $P(B|A^c) = P(BA^c)/P(A^c)$. We have:

   - $P(A^c) = 1 - P(A) = .6$.
   - $P(BA^c) = P(B) - P(BA) = P(B) - (P(A) + P(B) - P(A \cup B)) = P(A \cup B) - P(A) = .7 - .4 = .3$.
   - So $P(B|A^c) = .3/.6 = 1/2$.

7. We have a wheel of fortune with the numbers from 1 to n on it. Perform the following experiment: Spin the wheel repeatedly and record the numbers that are selected. Continue spinning as long as all of the numbers that have been selected are different. Stop the first time the number that is selected repeats a previously selected number. Let X be the random variable which records the total number of spins.

   (a) (**3 points**) Compute the probability that $X \geq 2$

   Solution. $P(X \geq 2) = 1 - P(X = 1) = 1$, since according to the rules we never stop after the first spin.

   (b) (**3 points**) Compute the probability that $X \geq 3$.

   Solution. We will spin a third time provided that the first two numbers are different. The 2nd spin will be the same as the first with probability $1/n$ so $P(X \geq 3) = 1 - 1/n$.

   (c) (**6 points**) What is the probability that $X = 5$?

   Solution. We will stop after the fifth spin, if the first four spins are different (which happens with probability $(\frac{n-1}{n})(\frac{n-2}{n})(\frac{n-3}{n})$) and the last spin matches one of the first four (which happens with probability $\frac{4}{n}$. So

   $$P(X = 5) = \frac{4(n-1)(n-2)(n-3)}{n^4}.$$

   (d) (**6 points**) Give the probability mass function for X.

   Solution. The possible values for X are $2, 3, \ldots, n+1$ since once we reach spin $n+1$ there must be a repeat. For $X = k$ to happen we must have that the first $k-1$ spins are different (which happens with probability $(\frac{n-1}{n})(\frac{n-2}{n}) \cdots (\frac{n-k+2}{n})$), and then the kth spin matches one of the first $k-1$ (which happens with probability $\frac{k-1}{n}$. So for $k \in \{2, \ldots, n+1\}$,

   $$P(X = k) = \frac{(k-1)(n-1)(n-2)\cdots(n-k+2)}{n^{k-1}}.$$