Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 9 [1]

(1) The joint density of X, Y is given by $f(x,y) = \frac{x}{2} + \frac{y}{4}$ for $0 \le x \le 1$ and $0 \le y \le 2$.

(a) Determine the marginal distribution of X.
Solution. For $x < 0$ or $x > 1$, $f_X(x) = 0$. For $x \in [0,1]$:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^2 (\frac{x}{2} + \frac{y}{4})dy \\ &= \frac{xy}{2} + \frac{y^2}{8}|_{y=0}^{y=2} = x + \frac{1}{2}. \end{aligned}$$

The distribution function is then $F_X(x) = \int_{-\infty}^x f(t)dt$. For $x < 0$ this is 0 (since $f(t) = 0$ for $t < 0$. For $0 < x \le 1$, this is $\int_0^x (t+1/2)dt = \frac{1}{2}t^2 + \frac{1}{2}t|_0^x = \frac{x^2+x}{2}$.
Note that when $x = 1$, this is 1, and for $x > 1$, $F_X(x) = 1$.

(b) Determine the conditional distribution of X given $Y = 1/2$.
Solution. First we compute the marginal density. We have $f_{X|Y=1/2}(x) = 0$ if $x < 0$ or $x > 1$. For $x \in [0,1]$, $f_{X|Y=1/2}(x) = f(x,1/2)/f_Y(1/2)$. The denominator is:

$$\begin{aligned} f_Y(1/2) &= \int_{-\infty}^{\infty} f(x,1/2)dx = \int_0^1 (\frac{x}{2} + \frac{1}{8})dx \\ &= \frac{x^2}{4} + \frac{x}{8}|_0^2 = \frac{3}{8} \end{aligned}$$

The numerator is $\frac{x}{2} + \frac{1}{8}$.
So the conditional density is: $f_{X|Y=1/2}(x) = (4x+1)/3$
The distribution function is 0 for $x < 0$, is $\int_{0^x} dt \frac{4t+1)}{3} = \frac{2x^2+x}{3}$ for $0 < x < 1$

(2) Let X_1, X_2, X_3 be independent discrete random variables that are each uniform over the set $\{1,2,3\}$.

(a) Determine the probability mass function of $S = X_1 + X_2 + X_3$.
Solution. The sample space is all $3^3 = 27$ sequences (x_1, x_2, x_3) where x_1, x_2, x_3 take on values in $\{1,2,3\}$. All 27 outcomes are equally likely. The possible values of S are $3,4,5,6,7,8,9$.

- $S = 3$ if and only if $(X_1, X_2, X_3) = (1,1,1)$, so $P(S = 3) = 1/27$.
- $S = 4$ if and only if one of (X_1, X_2, X_3) is 2 and the other two are 1. There are 3 ways to choose the 2, so there are 3 such outcomes, so $P(S = 4) = 3/27 - 1/9$.
- $S = 5$ if and only if one of (X_1, X_2, X_3) is 3 and the other two are 1 (three ways for this to happen) or one of them is 1 and the other two are 2 (three ways for this to happen). Thus $P(S = 5) = 6/27 = 2/9$.
- $S = 6$ if and only if (X_1, X_2, X_3) is a permutation of $1,2,3$ (6 ways), or is $(2,2,2)$. Thus $P(S = 6) = 7/27$.
- $S = 7$ if and only if one of (X_1, X_2, X_3) is 3 and the other two are 2 (three ways for this to happen) or one of them is 1 and the other two are 3 (three ways for this to happen). Thus $P(S = 5) = 6/27 = 2/9$.
- $S = 8$ if and only if one of (X_1, X_2, X_3) is 3 and the other two are 2. There are 3 ways to choose the 2, so there are 3 such outcomes, so $P(S = 4) = 3/27 - 1/9$.

[1] Version: 12/7/13

- $S = 9$ if and only if $(X_1, X_2, X_3) = (3, 3, 3)$, so $P(S = 9) = 1/27$.

(b) Determine probability mass function of (X_1, X_2, X_3) given that $S = 6$.
Solution. As given above, given that $S = 6$ we have (X_1, X_2, X_3) is either $2, 2, 2$ or is a permutation of $(1, 2, 3)$. So when conditioned on $S = 6$, each of these outcomes has probability $1/7$.

(c) Determine the marginal probability mass function of X_1 conditioned on $S = 6$.
Solution. Conditioned on $S = 6$, there are two outcomes, $(1, 2, 3)$ and $(1, 3, 2)$ where $X_1 = 1$, three outcomes, $(2, 1, 3)$ and $(2, 3, 1)$ and $(2, 2, 2)$ where $X_1 = 2$ and two outcomes, $(3, 1, 2)$ and $(3, 2, 1)$, where $X_1 = 3$. Thus:
$P(X_1 = 1|S = 6) = 2/7$, $P(X_1 = 2|S = 6) = 3/7$ and $P(X_1 = 3|S = 6) = 2/7$.

(3) Let $X_1, X_2, \ldots, X_{10}$ be independent random variables each having a normal distribution with mean 10 and variance 20.

(a) Estimate the probability that $X_1 \geq 12$.
Solution. $P(X_1 \geq 12) = 1 - \Phi((12 - 10)/\sqrt{20}) \approx 1 - \Phi(.447) \approx .328$.

(b) Let Y be the average of X_1 and X_2. Estimate the probability that $Y \geq 12$.
Solution. Here (and in the next problem) we use two basic facts about normal random variables The sum of independent normal random variables is normal with mean equal to the sum of the means and variance equal to the sum of the variances. Also if W is a normal random variable and C is any constant, then CW is a normal random variable with mean $C\mathbb{E}(W)$.
Applying this, Y is normally distributed with mean $\frac{1}{2}(\mathbb{E}[X_1]+\mathbb{E}[X_2]) = 10$ and variance $(\frac{1}{2})^2(Var(X_1) + Var(X_2)) = 10$.
So $P(Y \geq 12) = 1 - \Phi((12 - 10)/\sqrt{10}) \approx 1 - \phi(.632) \approx .263$

(c) Let Z be the average of $X_1, \ldots, X_{10}$. Estimate the probability that $Z \geq 12$.
Solution. Z is normally distributed with mean $\frac{1}{10}(\sum_{i=1}^{1} 0\mathbb{E}[X_i] = 10$ and variance $(\frac{1}{10})^2(\sum_{i=1}^{1} 0Var(X_i) = 2$.
So $P(Y \geq 12) = 1 - \Phi((12 - 10)/\sqrt{2}) \approx 1 - \phi(1.414) \approx .079$

(d) (Not to be graded). The above results illustrate a general principle involving averages of independent identically distributed normal random variables. Try to formulate such a principle.
Solution. The average A_n of n independent identically distributed normal random variables with mean μ and variance v, is normally distributed with mean μ and variance v/n. For any fixed number $b > \mu$, the probability that $A_n > b$ is $1-\Phi(\sqrt{n}(b-\mu)/\sqrt{v})$. As n gets larger the argument a of Φ gets larger so $\Phi(a)$ gets closer to 1 and the probability that $A_n > b$ gets closer to 0. A similar thing would happen if $b < \mu$ and we ask for the probability that $A < \mu$. Thus as n gets larger the average of n samples from $N(\mu, v)$ becomes more and more concentrated around μ.

(4) A particular rare disease affects men at a different rate than women. In a given year, the probability that a given man contracts the disease is $1/600,000$ while the probability that a given woman contracts the disease is $1/400,000$. For a city with 1,000,000 men and 1,000,000 women, estimate the probability that there are more than 6 cases of the disease.

Solution. The number of men who contract the disease is binomially distributed with $n = 1,000,000$ and $p = 1/600,000$. Since $np = 5/3$ is small, we can approximate the number of men who get the disease by a Poisson random variable with parameter $5/3$.

Similarly we can approximate the number of women who get the disease by a Poisson random variable with parameter $1,000,000/400,000 = 5/2$.

The number of people who contract the disease is then the sum of two Poisson random variables, one with parameter 5.3 and the other with parameter $5/2$. Assuming that the number of men who contract the disease is independent of the number of women (which may or may not be a reasonable assumption), this number is distributed like a Poisson with parameter $5/3 + 5/2 = 25/6$.

So the probability of more than 6 cases of the disease is 1 minus the probability of 6 or fewer cases which is approximated by:

$$1 - \sum_{i=0}^{6} (25/6)^i e^{-25/6}/i! \approx 0.128$$

(5) Suppose X and Y have joint density function $f(x, y)$ which is given by $\frac{6}{x^4 y^3}$ when $x \geq 1$ and $y \geq 1$ and is 0 otherwise.

(a) Suppose Z is the maximum of X and Y. Find the Cumulative distribution function for Z.

Solution. For $z < 1$ this is 0. For $zgeq1$ we get:

$$\begin{aligned}
P(Z \leq z) &= P(X \leq z \text{ AND} Y \leq z) \\
&= \int_{y=1}^{z} dy \int_{x=1}^{z} dx \frac{6}{x^4 y^3} \\
&= \int_{y=1}^{z} dy \frac{-2}{x^3 y^3} |_{x=1}^{x=z} \\
&= \int_{y=1}^{z} dy \frac{2}{y^3} (1 - \frac{-2}{z^3}) \\
&= \frac{-1}{y^2} (1 - \frac{-1}{z^3}) |_{y=1}^{y=z} \\
&= (1 - \frac{1}{z^2})(1 - \frac{1}{z^3}).
\end{aligned}$$

(b) (Not to hand in.) Find the expected value of Z.

Solution. We can use the solution to the first part and differentiate it to get the density function for Z and then use the definition of expectation to compute it. Alternatively, since Z is a nonnegative random variable, we can use the alternative formula $\int_0^\infty (1 - F_Z(z))dz$. This gives $\int_0^1 dz + \int_1^\infty (z^{-2} + z^{-3} - z^{-5})dz = 1 + (-z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{4}z^{-4})|_{z=1}^{z=\infty} = \frac{9}{4}$.