Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 7 [1]

(1) The following problem makes use of the table on page 190 of the book. Suppose that X is a Normal random variable with mean 5 and variance 10.
  (a) Estimate the probability that X is between 4 and 7.
  Solution. Since X is normally distributed with mean 5 and variance 10, $Y = (X-5)/\sqrt{10}$ is normally distributed with mean 0 and variance 1. Therefore $P[4 \leq X \leq 7] = P[\frac{4-5}{\sqrt{10}} \leq Y \leq \frac{7-5}{\sqrt{10}}] \approx P[-0.316 \leq Y \leq 0.632]$.
  Using the notation of the book $\Phi(x)$ is the Distribution function for Y. Hence the desired probability is approximately $\Phi(.632) - \Phi(-0.316)$. By the symmetry of the normal distribution $\Phi(-0.316) = 1 - \Phi(0.316)$, so using the table, the desired probability is approximately:

$$\Phi(.632) + \Phi(.316) - 1 \approx .732 + .624 - 1 \approx .356.$$

  (b) Estimate the endpoints of an interval I that is symmetric around the mean and is as small as possible such that $P[X \in I] \geq .8$.
  Solution. Using the notation from the solution to the first part, let Y be normal with mean 0 and variance 1. First we find the smallest possible interval J symmetric around 0 such that $P[Y \in J] \geq 0.8$. For this we want to determine a number a such that $P[-a \leq Y \leq a] \geq .8$. Now:

$$P[-a \leq Y \leq a] = \Phi[a] - \Phi(-a) = 2\Phi(a) - 1,$$

  so we want $\Phi(a) \geq .9$. Using the table we want a to be at least 1.282. Thus the desired interval is $[-1.282, 1.282]$.
  Now $X = 5 + \sqrt{10}Y$. So for any $b > 0$, $P[-b \leq Y \leq b] = P[5 - b\sqrt{10} \leq Y \leq 5 + b\sqrt{10}]$ Therefore the desired interval is (approximately) $5 - 1.282\sqrt{10}, 5 + 1.282\sqrt{10}] \approx [.946, 9.054]$.

(2) Suppose X is an exponential random variable with mean 3. Suppose $Y = e^X$. Find the pdf for Y.
  Solution. Let f be the pdf for X and F be the CDF for X. Let h be the pdf for Y and H be the CDF for Y. Since the mean of an exponential random variable is $1/\lambda$, $f(x) = \frac{1}{3}e^{-x/3}$ for $x \geq 0$ and is 0 for $x < 0$. Therefore the CDF satisfies $F(x) = 1 - e^{-x/3}$ for $x \geq 0$ and $F(x) = 0$ for $x < 0$. Since $Y = e^X$ and $P[X < 0] = 0$ we also have $P[Y < 1] = P[e^X < 1] = P[X < 0] = 0$. For $y > 1$, $P[Y \leq y] = P[X \leq \ln(y)]$ which is $1 - e^{-\ln(y)/3} = 1 - 1/y^{1/3}$. Therefore $H(y) = 0$ for $y < 1$ and is $1 - 1/y^{1/3}$ for $y > 1$. Finally $h(y) = \frac{dH}{dy}$ which is 0 for $y < 1$ and is $\frac{1}{3}y^{-2/3}$ for $y \geq 1$.

(3) If we roll a fair die 1000 times.
  (a) Use the normal approximation (and the table on page 190 of the book) to estimate the probability that 6 comes up at least 200 times.
  Solution. Let X denote the number of sixes rolled. The expected number of 6's is $1000(1/6)$ and the variance of the number of 6's is $1000(1/6)(5/6)$ so the number of 6's that come up is approximately normally distributed with mean $1000(1/6) \approx 167$ and variance $1000(1/6)(5/6) = 1250/9$. Therefore $Y = (X - 167)/\sqrt{1250/9}$ is approximately normally distributed with mean 0 and variance 1, and using the continuity

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correction:

$$\begin{aligned} P[X \geq 200] &\approx P[Y \geq (199.5 - 166.7)/25\sqrt{2}/3] \approx 1 - \Phi((199.5 - 166.7)3/25\sqrt{2}) \\ &\approx 1 - \Phi(2.78) \approx 1 - .9973 = .0027 \end{aligned}$$

(b) Approximately how many times do we need to roll the die in order that the following holds: The probability that the number of 6's rolled exceeds the expected number of 6's by more than 5% is less than 1/20.

Solution. Let n denote the number of rolls and X be the random variable that is tne number of 6's. We want to determine n so that the above conclusion holds. The expected number of 6's is $n/6$ and the variance is $5n/36$. Five percent more than the expected number of 6's is $1.05n/6$. We want to choose n so that $P[X \geq 1.05n/6]$ is at most 1/20.

X is approximately normally distributed with mean $n/6$ and variance $5n/36$. So $Z = (X - n/6)/\sqrt{5n/36}$ is normally distributed with mean 0 and variance 1. The condition $X \geq 1.05n/6$ is equivalent to $Z \geq (1.05n/6 - n/6)/\sqrt{5n/36}$ and we can rewrite this condition as $Z \geq .05\sqrt{n}/\sqrt{5} \approx .02236\sqrt{n}$. The probability that $Z \geq .02236\sqrt{n}$ is $1 - \Phi(.02236\sqrt{n})$ and we want that to be at most 1/20 so we want $\Phi(.02236\sqrt{n}) \geq .95$. Using the table we see that $\Phi(x) \geq .95$ provided that $x \geq 1.645$. So we want $.02236\sqrt{n} \geq 1.645$ which means we need n to be at least 5412 (approximately).

(4) The distribution of the lifetime of a particular manufactured product has hazard rate function $\lambda(t) = t^3/9$ for $t > 0$. Determine:

(a) The probability that the item survives for at least 3 years.

Solution. Let X be the random variable reprenting the lifetime. Using equation (5.4) in the book, the distribution function of X is given by:

$$\begin{aligned} F(t) &= 1 - exp\left(-\int_0^t x^3/6dx\right) \\ &= 1 - e^{-t^4/36}. \end{aligned}$$

So $P[X \geq 3] = 1 - F(3) = e^{-81/36} \approx .105$.

(b) The probability that the item survives for between 2 and 3 years.

Solution. Using $F(t)$ from the first part we get:
$P(2 \leq X \leq 3) = F(3) - F(2) = (1 - e^{-81/36}) - (1 - e^{-16/36}) = e^{-16/36} - e^{-81/36} \approx .536$.

(c) The conditional probability that the item survives for at least 3 years given that it survived at least 2 years.

Solution. $P(X \geq 3|X \geq 2) = P(X \geq 3)/P(X \geq 2) = e^{-81/36}/e^{-16/36} = e^{-65/36} \approx 0.164$.

(5) Suppose we have a polynomial function $p(x) = x^2 - Ax + 2A - 3$, where A is an unspecified constant. Suppose that A is chosen uniformly at random between 0 and 10. What is the probability that the function $p(x)$ is strictly positive for all x? (Hint: First determine conditions on the constants c and d such that the polynomial $x^2 + cx + d$ is positive for all x.)

Solution. A function of the form $x^2 + cx + d$ for constants c and d is strictly positive for all x if its minimum value is positive. The minimum occurs where the derivative is 0, which is at $x = -c/2$. Plugging this in gives that the minimum is $d - c^2/4$. So the minimum is strictly positive if $4d > c^2$.

In this case we have $d = 2A - 3$ and $c = -A$ so we need $8A - 12 \geq A^2$ or $A^2 - 8A + 12 < 0$ which is $(A - 6)(A - 2) < 0$. This holds if $2 < A < 6$. Since A is uniform between 0 and 10, this holds with probability 4/10.