Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 6 [1]

(1) Suppose that X is a random variable whose density is given by $f(x) = 0$ for $x < 0$, $f(x) = Ce^{-2x}$ for $0 \le x \le 2$ and $f(x) = Ce^{-(x+2)}$ for $x > 2$, where C is some constant.
   (a) Determine C.
   Solution. We know that $\int_{-\infty}^{\infty} f(x) = 1$. In this case

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) &= \int_0^2 Ce^{-2x} + \int_2^{\infty} Ce^{-(x+2)} \\ &= \frac{-C}{2} e^{-2x}|_0^2 - e^{-(x+2)}|_2^{\infty} \\ &= \frac{C}{2}(1 - e^{-4} + Ce^{-4}) = \frac{C}{2}(1 + e^{-4}). \end{aligned}$$

   So $C = 2/(1 + e^{-4})$.
   (b) Find the CDF for X.
   Solution. The CDF is given by $F(x) = \int_{-\infty}^{x} f(t)dt$. This is 0 for $x < 0$. For $x \in [0, 2]$:

$$F(x) = \int_0^x f(t)dt = \frac{C}{2}(1 - e^{-2x}) = \frac{1 - e^{-2x}}{1 + e^{-4}}.$$

   For $x > 2$,

$$\begin{aligned} F(x) &= \int_0^2 f(t)dt + \int_2^x f(t)dt \\ &= \frac{1 - e^{-4}}{1 + e^{-4}} + \frac{2}{1 + e^{-4}}(-e^{-(x+2)} + e^{-4}) \\ &= \frac{1 + e^{-4} - e^{-(x+2)}}{1 + e^{-4}}. \end{aligned}$$

(2) (This problem counts as two usual problems (10 points)) Let X be a continuous random variable whose probability density function has the form:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{Bx}{A} & \text{if } 0 \le x \le A \\ B & \text{if } A < x \le A + C \\ 0 & \text{if } x > A + C \end{cases}$$

   where A, B, C are undetermined positive constants.
   (a) Sketch the graph of the density function. Show the shape of the graph and label the (x, y) coordinates of key points (these labels will depend on A, B, C).
   [Solution not provided.]
   (b) Use the fact that the definite integral of the PDF is 1 to determine an equation relating A, B, C.

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Solution.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_0^A \frac{B/A}{x}dx + \int_A^{A+C} Bdx \\ &= \frac{B}{A}\frac{x^2}{2}|_0^A + Bx|_A^{A+C} \\ &= \frac{AB}{2} + BC = B(C + \frac{A}{2}).\end{aligned}$$

Therefore $B(C + \frac{A}{2}) = 1$.

(c) Express the expected value of X in terms of A, B, C.

Solution.

$$\begin{aligned}E[X] &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^A \frac{B/A^2}{x} dx + \int_A^{A+C} Bxdx \\ &= \frac{B}{A}\frac{x^3}{3}|_0^A + B\frac{x^2}{2}|_A^{A+C} \\ &= \frac{A^2B}{3} + \frac{B}{2}(2AC + C^2) = B(\frac{A^2}{3} + AC + \frac{C^2}{2}).\end{aligned}$$

(d) Express the probability that $X \geq A + C/2$ in terms of A, B, C.

Solution. Notice that $A + C/2$ is in the interval between A and $A + C$. The desired probability is given by:

$$\begin{aligned}\int_{A+C/2}^{A+C} f(x)dx &= \int_{A+C/2}^{A+C} Bdx \\ &= Bx|_{A+C/2}^{A+C} = BC/2.\end{aligned}$$

(e) Given that $E[X] = 71/24$ and $Prob[X \geq A + C/2] = 3/8$ determine A, B and C.

Solution. From part (d) and the fract that $Prob[X \geq A + C/2] = 3/8$ we have $BC/2 = 3/8$ so $BC = 3/4$ so $B = \frac{3}{4C}$.

From part (b) we know that $B = 2/(A + 2C)$ so $\frac{3}{4C} = \frac{2}{A+2C}$ which simplifies to $3A = 2C$ or $A = \frac{2C}{3}$.

We can use these to substitute into the expression in part (c) for $E[X]$ to express $E[X]$ in terms of C:

$$\begin{aligned}E[X] &= B(\frac{A^2}{3} + AC + \frac{C^2}{2} \\ &= \frac{3}{4C}(\frac{4}{27}C^2 + \frac{2}{3}C^2 + \frac{1}{2}C^2) \\ &= \frac{3}{4C}(\frac{4}{27}C^2 + \frac{2}{3}C^2 + \frac{1}{2}C^2) = \frac{213}{216}C.\end{aligned}$$

We are given that this equals $71/24$ and so $C = 3$. Substituting into the expressions for A and B we get $A = 2$ and $B = 1/4$.

(3) Suppose we choose a ray starting at the origin which makes an angle θ with the positive real axis, where θ is chosen uniformly between 0 and π. Consider the point (X, Y) where the ray hits the unit circle.

(a) Find $E[X]$.

Solution. Let $f(t)$ be the density function for θ. Since θ is uniform on $[0,\pi]$, $f(t) = 1/\pi$ for $0 \le t \le \pi$ and $f(t) = 0$ otherwise.
Notice that $X = \cos(\theta)$ and $Y = \sin(\theta)$.

$$\begin{aligned} E(X) &= \int_0^\pi f(\theta)\cos(\theta)d\theta \\ &= \frac{1}{\pi}\sin(\theta)|_0^\pi = 0. \end{aligned}$$

(b) Find $E[Y]$.
Solution. Similar to the previous part we get:

$$\begin{aligned} E(Y) &= \int_0^\pi f(\theta)\sin(\theta)d\theta \\ &= \frac{1}{\pi} - \cos(\theta)|_0^\pi = \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}. \end{aligned}$$

(c) Find $Var(X)$.
Solution. $Var(X) = E[X^2] - E[X]^2$ which is just $E[X^2]$, by part (a). So $Var(X) = E[X^2] = \frac{1}{\pi}\int_0^\pi (\cos(\theta))^2 d\theta$.
Now we need to remember how to integrate $(\cos(\theta))^2$. Recall that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1$ so $\cos^2(\theta) = \frac{1}{2} + \frac{\cos(2\theta)}{2}$. So:

$$\begin{aligned} E[X^2] &= \frac{1}{\pi}\int_0^\pi (\frac{1}{2} + \frac{\cos(2\theta)}{2})d\theta \\ &= \frac{1}{\pi}(\frac{\theta}{2} + \frac{\sin(2\theta)}{4})|_0^\pi \\ &= \frac{1}{\pi}(\frac{\pi}{2} = \frac{1}{2}. \end{aligned}$$

(4) Suppose that X and Y are independent random variables both having pdf $f(x) = e^{-x}$.

(a) Let Z be the maximum of X and Y. Find the pdf for Z. (Hint: Notice that for any number t, $P[Z \le t] = P[(X \le t) \text{ AND}(Y \le t)]$. Use this and the fact that X and Y are independent to find the CDF of Z.
Solution. Let g be the pdf for Z and G be the CDF. We will find G and use it to find g. Let F be the CDF for X and Y. By the hint $G(t) = P[Z \le t] = P[(X \le t) \text{ AND } (Y \le t)]$. Since X and Y are independent, this is equal to $P[X \le T]P[Y \le t] = F(t)^2$. Since , $F(t) = \int_0^t f(x)dx = 1 - e^{-t}$. we obtain $G(t) = (1-e^{-t})^2 = 1 - 2e^{-t} + e^{-2t}$. Finally $g(t) = \frac{d}{dt}G(t) = 2(e^{-t} - e^{-2t})$

(b) Let W be the minimum of X and Y. Find the pdf of W.
Solution. Let h be the pdf for W and H be the CDF. We will find H and use it to find h. Let F be the CDF for X and Y. We have

$$\begin{aligned} H(t) &= P[W \le t] = P[(X \le t) \text{ OR } (Y \le t)] \\ &= P[X \le t] + P[Y \le t] - P[(X \le t) \text{ AND } (Y \le t)] \\ &= 2(1-e^{-t}) - (1-e^{-t})^2 = 1 - e^{-2t}. \end{aligned}$$

Therefore $h(t) = \frac{d}{dt}H(t) = 2e^{-2t}$.