Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 5 [1]

(1) A manufacturer makes a certain electronic part. The probability that a given part is malfunctioning is 1/500. The manufacturer can do a test on a group of parts. The group will pass the test if all of the parts in the group are functioning and will fail the test if one or more parts in the group is malfunctioning. Suppose we test a group of 200 parts.
   (a) Assuming the assumptions of the problem are correct, what is the type of random variable and appropriate parameters exactly represents the number of malfunctioning parts in a group?
   Solution. Assuming that each part functions or malfunctions independently, the distribution is binomial with parameters 200 and 1/500.
   (b) What random variable (and with what parameter(s)) can be used to give a good estimate to the random variable in the previous part?
   Solution. A binomial with parameters n and p can be estimated by a Poisson with parameter $\lambda = np$ provided that p is small and n is large. So we can estimate the distribution by a Poisson with parameter .4.
   (c) Give a decimal estimate for the probability that the group fails the test.
   Solution. Let X be the number of failures in a group. A group fails the test if $X \geq 1$ so it fails with probability $1 - P(X = 0)$. Using the Poisson approximation this is about $1 - e^{-.4} \approx .33$.
   (d) If the group fails the test, estimate the conditional probability that there is exactly one part in the group that is malfunctioning.
   Solution. We want $P(X = 1|X \geq 1) = P(X = 1 \text{ AND} X \geq 1)/P(X \geq 1) = P(X = 1)/P(X \geq 1) \approx .4e^{-.4}/(1 - e^{-.4}) \approx .813$

(2) A salesperson makes calls to potential customers. The probability that a given call results in a sale is 2/7. Suppose the salesman stops for the day after making 6 sales.
   (a) What random variable (and with what parameters) represents the total number of calls made by the salesman?
   Solution. This is like a coin flipping process where we want to known how many coin flips until we get 6 heads. This is a negative binomial random variable with parameters 6 and 2/7.
   (b) What is the probability that the number of calls made is more than 20? (You may express this as a sum.)
   Solution. The salesman will make more than 20 calls if the first 20 calls result in at most 5 sales. The number of sales in 20 calls is a binomial random variable with parameters 20 and 2/7, so the probability of at most 5 sales is equal to $\sum_{i=0}^{5} \binom{20}{i} (2/7)^i (5/7)^{20-i}$.
   (c) What is the expected number of calls?
   Solution. The expected value for a negative binomial with parameters r and p is r/p so this is $6/(2/7) = 21$.
   (d) What is the variance of the number of calls?
   Solution. The variance of a negative binomial with parameters r and p is $r(1-p)/p^2$ which in this case is $6(5/7)/(2/7)^2 = 52.5$

(3) Consider a gambling game in which a player selects 5 different numbers between 1 and 10, and then 5 random numbers are selected (without replacement) between 1 and 10. The payoffs are as follows: If the player matches all 5 of the selected numbers he wins \$400, if the player matches 4 numbers he wins \$20 and if he matches 0 of the selected numbers he wins \$200. If it costs \$5 to play and X is the amount won minus the cost of playing, find

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the expected value of X. (The random variable X is called the *payoff* of the game so you are computing the expected payoff of the game.)

Solution. Once the player fixes his 5 numbers, think of the selected numbers as "white" and the unselected numbers as "red". When 5 random numbers are chosen, let X be the number of the players numbers that are chosen. The probability $X = i$ is $\binom{5}{i}\binom{5}{5-i}/\binom{10}{5}$ (This is a hypergeometric random variable with 10 balls total, of which 5 are white, and 5 balls are selected without replacement). The expected amount won (not counting the cost of playing is:

$$400P(X=5) + 200P(X=0) + 20P(X=4) = (400 + 200 + 20(25))/\binom{10}{5} = 1100/252.$$

Subtracting off the cost of the ticket gives expected payoff of $1100/252 - 5 = -160/252 \approx -.63$.

(4) A group of 6 players numbered 1 to 6 play a sequence of games. In each game there is one winner, and p_i is the probability that player i wins a given game. If the players play exactly 6 games, what is the expected number of players who win no games. (Your answer will be a sum involving the quantities $p_1, p_2, p_3, p_4, p_5, p_6$.)

Solution. Define X to be the number of players who win no games. Computing the expected value of X directly can be done but is very tedious because its hard to compute $P(X = i)$ for different values of i. So instead we'll use the method taught in class (and used in problem 3d of the exam): Express X as a sum of simpler random variables and compute $E[X]$ by adding up the expectation of the simpler random variables.

So we have to come up with a clever way to write X as a sum of simpler random variables. For i between 1 and 6, define X_i to be the RV which is 1 if player i wins no games and 0 otherwise. Then $X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$. Since X_i is Bernoulli $E[X_i] = P[X_i = 1]$ which is $(1 - p_i)^6$ since this is the probability that i loses all 6 games. So $E[X] = \sum_{i=1}^{6}(1 - p_i)^6$.

(5) You are going hunting for pearls. If you select a clam at random and open it, the probability that it contains a pearl is about 1 in 12000.

(a) Estimate the number of clams that must be opened so that the expected number of pearls found is 1.

Solution. Let X represent the number of clams opened before finding a pearl. X is a geometric random variable with probability $p = 1/12000$. So the expected number of clams that are opened is $1/p = 12000$.

(b) Estimate the number of clams that must be opened so that the probability of finding a pearl is at least 1/2.

Solution. We want to choose n so that if we look at n clams the probability of NOT finding a pearl is less than 1/2. Using the Poisson approximation with $\lambda = np = n/12000$, the probability of not finding a pearl in n clams is about $e^{-n/12000}$. To make this less than 1/2 we want $n \geq 12000 \ln(2) \approx 8318$.

(c) Estimate the number of clams that must be opened so that the probability of finding a pearl is at least .9

Solution. Similar to the previous part except now we want $e^{-n/12000} \leq .1$ so $n \geq 12000 \ln(10) \approx 27631$.