Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 4 [1]

Problems to be handed in.

(1) A student takes a multiple choice test with 5 questions. Each question has 4 possible answers. The test is scored by giving 3 points for a correct answer and -1 for an incorrect answer. Suppose that for each question, a given student is able to eliminate one obviously incorrect answer, and then randomly quesses one of the remaining answers. (a) Find the probabiility mass function for the student's score. (b) Compute the expected value (c) Compute the variance of the student's score. (There is a simpler way to compute the expected value and variance than just applying the definition; see if you can find it.)

Solution. (a) Let R be the number of right answers and W be the number of incorrect answers. Notice that $W = 5 - R$ so the total score is $S = 3R - W = 4R - 5$.

We want the PMF of S which is a function we'll denote by f. The domain of the function is the set of possible score values which is the set $PSV = \{-5, -1, 3, 7, 11, 15\}$.

Now it's not so easy to see how to figure out the PMF for S, but it is easy to get the PMF for R, which we'll call g. On each question the student is randomly guessing among the 3 answers he couldn't eliminate, so the number R of right answers is like a coin-flipping process with 5 coin flips, each having probability of success 1/3, so this is a binomial random variable with parameters 5 and 1/3. So the PMF of R is given by $g(j) = P(R = j) = \binom{5}{j}(1/3)^j(2/3)^{5-j}$ for $j \in \{0, 1, 2, 3, 4, 5\}$.

Now we want to use g to figure out the function f. If k is a possible score value, then k corresponds to a number j of correct answers by the formula $k = 4r - 5$. Therefore given the score k we can write the number of correct answers as $r = (k + 5)/4$. Therefore for a possible score value $k \in PSV$,

$$f(k) = P(S = k) = P(R = (k+5)/4) = g((k+5)/4) = \binom{5}{(k+5)/4}(1/3)^{(k+5)/4}(2/3)^{5-(k+5)/4}.$$

(b) As mentioned $E[S]$ be done directly from the formula for expected value. But there are easier ways. Since $S = 4R - 5$ we can use Corollary 4.1 of the book to say that $E[S] = 4E[R] - 5$. So let's figure out $E[R]$ which is the expected number of correct answers. Since R is a binomial random variable with parameters 5 and 1/3 we can use the formula on page 162 to say that $E[R] = 5(1/3) = 5/3$. Therefor $E[S] = 4(5/3) - 5 = 5/3$.

(c) Again, we could compute $Var(S)$ from the definition of variance. Instead we'll use the identity at the bottom of page 162 (which says that for a random variable X and constants a and b, $Var(aX + b) = a^2Var(X)$ and therefore $Var(S) = Var(4R - 5) = 16Var(R)$. Since R is a binomial RV with parameters 5 and 1/3 we use the formula for variance of binomial random variables (page 132) to get that $Var(R) = (1/3)(2/3)5 = 10/9$ and so $Var(S) = 160/9$.

(2) A device is built from four components, each of which fails independently with probability p. The device functions properly provided that at least 3 of the components are working. (a) If 8 devices are built, what is the probability that at least 6 function properly? (b) What is the expected number of components that work properly?

Solution. For part (a): Let q denote the probability that a single device functions properly. We first compute q. A single device is built of 4 components each failing with probability p and working with probability $1 - p$. So the probability q that a device works is the probability that 0 components fail, or 1 components fails which is $q = \binom{4}{0}(1-p)^4 + \binom{4}{1}p(1-p)^3 = (1-p)^3(1+3p)$.

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Now we build 8 devices. Let X be the random variable which is the number of devices that work. Since each works with probability q (where q is given by the formula above), X is a binomial random variable with parameters 8 and q. Therefore the PMF of X is given by $P(X=j) = \binom{8}{j} q^j (1-q)^{8-j}$ for $j \in \{0, 1, \ldots, 8\}$, where q is given above.

For part (b): There are 8 devices with 4 components each, so 32 components all together. Each works with probability $1-p$ so the expected number of working components is $32(1-p)$.

(NOTE: I actually intended to ask a different question: "What is the expected number of devices that work properly?" In the previous version of the solutions I gave a solution to that problem: By the formula for expected value of a binomial random variable $E[X] = 8q = 8(1-p)^3(1+3p)$.)

(3) We have three coins. One has probability of heads equal to 1/4, one has probability of heads equal to 3/4 and one is a fair coin. We select a coin at random, and flip it four times. If the number of heads is two, find the conditional probability that we selected the fair coin. Express your answer as a fraction in lowest terms.

Solution. Define the following events:

- C_1 is the event that we chose the coin having probability of heads 1/4
- C_2 is the event that we chose the coin having probability of heads 1/2
- C_3 is the event that we chose the coin having probability of heads 3/4
- E is the event that we flipped exactly two heads.

We want $P(C_2|E)$ which equals $P(E|C_2)P(C_2)/P(E)$ by Bayes' rule. We have $P(C_2) = 1/3$ since we selected the coin at random from 3 coins. $P(E|C_2) = \binom{4}{2}(1/2)^2(1/2)^2 = 3/8$ since this is the probability of exactly 2 heads with a fair coin.

Finally,

$$\begin{aligned} P(E) &= P(E|C_1)P(C_1) + P(E|C_2)P(C_2) + P(E|C_2)P(C_3) \\ &= \binom{4}{2}(1/4)^2(3/4)^2(1/3) + \binom{4}{2}(1/2)^2(1/2)^2(1/3) + \binom{4}{2}(3/4)^2(1/4)^2 \\ &= 17/64. \end{aligned}$$

Putting it all together we get $P(C_2|E) = (1/8)/(17/64) = 8/17$.

(4) We have a coin whose probability of heads is p. Suppose we flip the coin until the number of heads seen is exactly 10. Let Y be the total number of coin flips. Find the probability mass function for Y. (Hint: notice that when we stop, the last coin flip must be heads.)

Solution. For each integer k we want $P(Y=k)$. For $Y=k$ we need that there were exactly 9 heads in the first $k-1$ flips and the final coin is heads. So $P(Y=k) = \binom{k-1}{9} p^9 (1-p)^{k-1-9} \times p = \binom{k-1}{9} p^{10}(1-p)^{k-10}$. (Note that if $k \leq 10$ then this is 0 since $\binom{a}{b} = 0$ whenever $b > a \geq 0$.:

(5) Suppose we have an urn with w white balls and r red balls. We select k balls without replacement from the urn (where k is a number less than or equal to $w+r$). Let X be the random variable which is the number of white balls in the sample. Find the probability mass function for X. (Hint: You may want to first try picking specific numbers for w, r and k and solving the problem for those numbers, and then generalize your solution to work for arbitrary w,r and k).

Solution. Let f be the PMF for X. The possible values for X are $0, 1, 2, 3, \ldots, w$. For $j \in \{0, 1, \ldots, w\}$ we have $f(j) = P(X=j)$. We select k balls out of $w+r$ so our sample space has size $\binom{w+r}{k}$. There are $\binom{w}{j}$ ways to choose exactly j white balls out of the w balls and $\binom{r}{k-j}$ ways to choose the remaining balls from the red balls. So

$$f(j) = P(X = j) = \frac{\binom{w}{j}\binom{r}{k-j}}{\binom{w+r}{k}}.$$

(Note that this is 0 if $j > w$ or $k - j > r$.)