# Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 3 [1]

- **Problem 2.52** Part (a) **Solution 1**. There are $\binom{20}{8}$ ways to select the shoes. The number of ways that contain no pair is the product of $\binom{10}{8}$ (the number of ways to choose which pairs are represented) times $2^8$ (the number of ways to choose one shoe from each selected pair). Thus the answer is $2^8 \times \binom{10}{8}/\binom{20}{8} \approx .09145$. **Solutions 2**. Let $E_i$ be the event that the $i$th chosen shoe doesn't match any previously selected shoe. Then we want $P(E_1E_2E_3E_4E_5E_6E_7E_8)$ which by the multiplication rule in chapter 3 is the product of $P(E_j|E_1 \cdots E_{j-1})$ for $j$ from 1 to 8. $P(E_j|E_1 \cdots E_{j-1} = \frac{n-2(j-1)}{n-(j-1)}$ since given that the first $j-1$ shoes are from different pairs, there are $n-j-1$ shoes remaining of which $n-2(j-1)$ don't match any already selected shoe. So the answer is $\frac{20}{20} \times \frac{18}{19} \times \frac{16}{18} \times \frac{14}{17} \times \frac{12}{16} \times \frac{10}{15} \times \frac{8}{14} \times \frac{6}{13} = \frac{12\times10\times8\times6}{19\times17\times15\times13} \approx .09145$

  Part (b). Using the approach in Solution 1 of part (a): The sample space has size $\binom{20}{8}$. The outcomes in which exactly one pair is selected can be counted as follows: Select the matched pair (10 choices) then select 6 of the remaining pairs (in $\binom{9}{6}$ ways) and select one shoe from each ($2^6$ choices). The resulting probability is then $2^6 \times 10 \times \binom{9}{6}/\binom{20}{8} \approx .4268$.
- **Problem 3.23** Let $W_1$ be the event that the ball selected from the first urn is white and $W_2$ be the event that the ball selected from the second urn is white. Part (a) By (3.1) of chapter 3, $P(W_2) = P(W_2|W_1)P(W_1) + P(W_2|W_1^c)|P(W_1^c)$. $P(W_1) = 2/6 = 1/3$ since Urn 1 has 2 white and 4 red, and $P(W_1^c) = 1 - P(W_1) = 2/3$. $P(W_2|W_1) = 2/3$ since given $W_1$ the second urn will have 2 white and 1 red, and $P(W_2|W_1^c) = 1/3$ since given $W_1^c$ the second urn will have 1 white and 2 red. Substituting those values into the earlier equation we have $P(W_2) = (2/3)(1/3) + (1/3)(2/3) = 4/9$.

  Part (b) By Bayes formula $P(W_1|W_2) = P(W_2|W_1)P(W_1)P(W_2)$ In part (a), we noted that $P(W_2|W_1) = 2/3$, $P(W_1) = 1/3$ and $P(W_2) = 4/9$ so we get $P(W_1|W_2) = (2/3)(1/3)/(4/9) = 1/2$.
- **Problem 3.32** Let $F_j$ be the event that the selected family has exactly $j$ children (for $j$ between 1 and 4). Let $E$ be the event that the selected child is the eldest in his or her family, and $Y$ be the event that the selected child is the youngest in his or her family.

  For part (a) we want to compute $P(F_1|E)$ which by Bayes' rule is $P(E|F_1)P(F_1)/P(E)$. $P(E|F_j) = 1/j$ since there is one eldest child out of $j$ children (given $F_j$). Using the values for $P(F_j)$ stated in the problem we get $P(E) = \sum_j P(E|F_j)P(F_j) = 1(.1) + (1/2)(.25) + (1/3)(.35) + (1/4)(.3) = 5/12$. Putting this together we get $P(E|F_1) = (1)(.1)/(5/12) = 12/50 = .24$.

  For part (b) we want to compute $P(F_4|E) = P(E|F_4)P(F_4)/P(E)$. $P(E|F_4) = 1/4$ and $P(F_4) = .3$ is given and we computed $P(E)$ within part (a) so we have $P(F_4|E) = (1/4)(.3)/(5/12) = .18$.

  If we condition on the child be the youngest instead of the eldest then we get the same answers in both cases since $P(Y|F_j) = P(E|F_j) = 1/j$ for each $j$.
- **Problem 3.78** For part (a). Let $E_j$ be the event that exactly $j$ games are played. We want $P(E_4)$. Let us use (3.1) of chapter 3 to write $P(E_4) = P(E_4|E_2)P(E_2) + P(E_4|E_2^c)P(E_2^c)$. Given $E_2$, $E_4$ can't happen so the first term in the sum is 0. For the second term: the series does not end after 2 games if $A$ wins one game and $B$ wins the other which happens with probability $2p(1-p)$. Given that the series did not end after 2 games, the series ends after 4 games if $A$ wins the next two or $B$ winse the next two which happens with probability $p^2 + (1-p)^2$ Thus we get $P(E_4) = 2p(1-p)(p^2 + (1-p)^2)$.

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For part (b), let $F$ be the event that $A$ wins. Let us condition on the outcome of the first two games.

These outcomes can be written as $AB$ ($A$ wins first $B$ wins the second), $BA$, $BB$, and $AA$.

$$\begin{aligned} P(F) &= P(F|AA)P(AA) + PP(F|BB)P(BB) + P(F|BA)P(BA) + P(F|AB)P(AB) \\ &= P(F|AA)p^2 + PP(F|BB)(1-p)^2 + P(F|BA)(1-p)p + P(F|AB)p(1-p). \end{aligned}$$

Given $AA$, $A$ has won already so $P(F|AA) = 1$. Given $BB$, $A$ has lost so $P(F|BB) = 0$. Given $AB$ happens the score is tied, so it is as though the series starts over so $P(F|AB) = P(F)$ and similarly $P(F|BA) = P(F)$. Substituting this information in we get $P(F) = p^2 + 2p(1-p)P(F)$, and solving for $P(F)$ we get $P(F) = p^2/(1-2p+2p^2)$.

- **Th. Ex. 3.13**

Let $E_{n,m}$ be the event that $A$ accumulates $n$ heads before $B$ accumulates $m$ tails. Let $P_{n,m} = P(E_{n,m})$. Important note: The probability that $A$ wins could change if $A$ went 2nd and $B$ went first. So $P_{n,m}$ is the probability that the first player to flip reaches $n$ heads before the second player to flip reaches $m$ heads.

Let $F$ be the event that the first coin (flipped by $A$) is heads. So $P(F) = p$. Then:

$$\begin{aligned} P_{n,m} = P(E_{n,m}) &= P(E_{n,m}|F)P(F) + P(E_{n,m}|F^c)P(F^c) \\ &= pP(E_{n,m}|F) + (1-p)P(E_{n,m}|F^c). \end{aligned}$$

Now if $F$ happens then $A$ has 1 head and needs $n-1$ more and $B$ still needs $m$ heads to win. Also it is still $A$'s turn to flip. Therefore the situation is the same as we started with except that $A$ only needs $n-1$ heads to win, so $P(E_{n,m}|F) = P_{n-1,m}$.

If $F^c$ happens then $A$ still needs $n$ heads and $B$ still needs $m$ heads to win. However, it is now $B$'s turn, so in the remaining game $B$ is the first player. So $P(B \text{ wins}|F^c) = P_{m,n}$. Notice that $m$ and $n$ switch places because $B$ is the first player now and his goal is $m$ heads. $P(A \text{ wins}|F^c) = 1 - P_{m,n}$.

Substituting in for our earlier expression for $P_{n,m}$ gives:

$$P_{n,m} = pP_{n-1,m} + (1-p)(1-P_{n,m}).$$