Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 2 [1]

- **Problem 3.4**. Let E be the event that at least one die is a 6 and S_i be the event that the sum of the 2 dice is i (for each i between 2 and 12). If $i \le 6$ then neither die can be 6 so $P(E|S_i) = 0$. For $i = 11$ or 12 one of the dice must be 6 so $P(E|S_i) = 1$. For the remaining cases $7 \le i \le 10$ we use the definition of conditional probability $P(E|S_i) = P(ES_i)/P(S_i)$. The numerator is always $2/36$ since the rolls $(6, i-6)$ and $(i-6, 6)$ are the only 2 rolls that satisfy. For the denominator, we count the number of rolls that sum to i (for $i = 7, 8, 9, 10$) and divide by 36. You can do this by listing the rolls, or you can reason as follows: If the dice sum to i then the two dice are $(j, i-j)$ where $j \ge i-6$ (since $i - j \le 6$) and $i \le 6$. There are $6-(i-6)+1 = 13-i$ integers between $i-6$ and 6. Therefore $P(S_i) = (13-i)/36$ and $P(E|S_i) = 2/(13-i)$ for $7 \le i \le 10$.
- **Problem 3.11**. Part 1. We use Using Bayes' formula $P(B|A_s) = P(A_s|B)P(B)/P(A_s)$, so we compute each of the three terms on the right. Our sample space has $\binom{52}{2}$ points (pairs of cards). There are $\binom{4}{2}$ ways to choose 2 aces out of 4, so $P(B) = \binom{4}{2}/\binom{52}{2} = 1/221$. To compute the other two terms, we use the fact that if we select k cards out of n, the chance that a specific card is included is k/n (which is easy to show; see example 5d of chapter 2). Therefore $P(A_s)$, the chance that 2 cards out of 52 include the ace of spaces, is $1/26$. Also $P(A_s|B) = 2/4 = 1/2$ since after conditioning on B we can view the sample space as select 2 cards out of the 4 aces. Thus $P(B|A_s) = \frac{1}{2} \times \frac{1}{221}/(1/26) = 1/17$.

  Part 2. Again we use Bayes formula: $P(B|A) = P(A|B)P(B)/P(A)$. We computed $P(B) = 1/221$ above. $P(A|B) = 1$ since given both cards are aces then certainly at least one ace was chosen. To compute $P(A)$ note this is 1 minus the chance that no Ace is chosen which happens with probability $\binom{48}{2}/\binom{52}{2} = 564/663$ so $P(A) = 99/663$. Thus we get $P(B|A) = (1/221)/(99/663) = 1/33$.
- **Problem 3.15**. Consider the probability space that consists of selecting a pregnant woman at random. Let E be the event that an ectopic pregnancy develops. Let S be the event that the woman is a smoker. We are asked to find $P(S|E)$. By Bayes' rule $P(S|E) = P(E|S)P(S)/P(E)$. We are given that $P(S) = .32$ and also $P(E|S) = 2*P(E|S^c)$. $P(E) = P(E|S)P(S) + P(E|S^c)P(S^c) = .32P(E|S) + .68P(E|S^c)$. Since $P(E|S) = 2P(E|S^c)$ this is equal to $.32P(E|S).68P(E|S)/2 = .66P(E|S)$. Substituting into the formula obtained by Bayes' rule we get: $P(S|E) = P(E|S)(.32)/(.66P(E|S)) = .32/.66 = \frac{16}{33}$.
- **Problem 3.40**. For $i = 1, 2, 3$ let W_i be the event that the ith selected ball is white. Let X be the number of selected white balls.
  - $P(X = 0) = P(W_1^c W_2^c W_3^c) = P(W_1^c)P(W_2^c|W_1^c)P(W_3^c|W_1^c W_2^c)$. $P(W_1^c) = 7/12$, since initially 7 out of 12 balls are not white. Given W_1^c we have 8 red and 5 white balls before the 2nd ball is selected so $P(W_2^c|W_1^c) = 8/13$. Given $W_1^c W_2^c$ we have 9 red and 5 white balls before the third ball is selected so $P(W_3^c|W_1^c W_2^c) = 9/14$. Combining we get:

    $$P(X = 0) = \frac{7 \times 8 \times 9}{12 \times 13 \times 14} = \frac{3}{13}.$$
  - $P(X = 1) = P(W_1 W_2^c W_3^c) + P(W_1^c W_2 W_3^c) + P(W_1 W_2^c W_3^c)$, where:

$$P(W_1 W_2^c W_3^c) = P(W_1)P(W_2^c|W_1)P(W_3^c|W_1 W_2^c) = \frac{5}{12} \times \frac{7}{13} \times \frac{8}{14} = \frac{5 \times 7 \times 8}{12 \times 13 \times 14}.$$

$$P(W_1^c W_2 W_3^c) = P(W_1^c)P(W_2|W_1^c)P(W_3^c|W_1^c W_2^c) = \frac{7}{12} \times \frac{5}{13} \times \frac{8}{14} = \frac{5 \times 7 \times 8}{12 \times 13 \times 14}.$$

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$$P(W_1^cW_2^cW_3) = P(W_1^c)P(W_2^c|W_1^c)P(W_3|W_1^cW_2^c) = \frac{7}{12} \times \frac{8}{13} \times \frac{5}{14} = \frac{5 \times 7 \times 8}{12 \times 13 \times 14}.$$

Adding these up we get $P(X = 1) = 3 \times \frac{5\times7\times8}{12\times13\times14} = \frac{5}{13}$.

- Following a similar reasoning as in the first part we have $P(X = 3) = P(W_1W_2W_3) = P(W_1)P(W_2|W_1)P(W_3|W_1W_2) = \frac{5}{12}\frac{6}{13}\frac{7}{14} = \frac{5}{52}$.
- $P(X = 2) = 1 - P(X = 0) - P(X = 1) - P(X = 3) = 1 - \frac{3}{13} - \frac{5}{13} - \frac{5}{52} = \frac{15}{52}$.

- **Problem 3.47**. For i between 1 and 6 let R_i be the event that the die roll is i. Let W be the event that all selected balls are white.

  To compute $P(W)$ we note that the events $D_1, D_2, D_3, D_4, D_5, D_6$ are mutually exclusive events which together cover the whole sample space. Using (3.4) from Chap. 4:

$$P(W) = \sum_i P(W|D_i)P(D_i).$$

  We have $P(D_i) = 1/6$, and $P(W|D_i)$ is the chance that if we select exactly i balls they will all be white which is $\binom{5}{i}/\binom{15}{i}$. This gives:

$$P(W) = \frac{1}{6}\sum_{i=1}^{6}\frac{\binom{5}{i}}{\binom{15}{i}} = \frac{5}{66}$$

  For the second part, we have

$$P(D_3|W) = \frac{P(D_3W)}{P(W)} = \frac{P(D_3)P(W|D_3)}{P(W)} = \frac{1}{6}\frac{\binom{5}{3}/\binom{15}{3}}{5/66} = \frac{22}{455}.$$

- **Th. Ex. 3.4**. Assume we search for the ball in box i. Let F_i be the event that we don't find it. Let E_j be the event that the ball is in box j. By Bayes' theorem $P(E_j|F_i) = P(F_i|E_j)P(E_j)/P(F_i)$. We compute each of the three terms on the righthand side:
  - We have $P(F_i|E_j) = 1$ if $i \neq j$ (since we definitely won't find the ball in box i if its in box j) and we are given that $P(F_i|E_i) = 1 - \alpha_i$.
  - We are given $P(E_j) = P_j$.
  - By Chap. 3 (3.4), $P(F_i) = \sum_j P(F_iE_j) = \sum_j P_jP(F_i|E_j)$. For all $i \neq j$ we have $P(F_i|E_j) = 1$ since we definitely won't find the ball in box i given it's in box j. Also $P(E_i)P(F_i|E_i) = P_i(1 - \alpha_i) = P_i - \alpha_iP_i$. Substituting into the sum we get

$$P(F_i) = P_i - \alpha_iP_i + \sum_{j\neq i} P_j = \sum_{j=1}^{n} P_j - \alpha_iP_i = 1 - \alpha_iP_i,$$

    where the last equality holds since $\sum_j P_j = 1$.

Substituting these into our expression for $P(E_j|E_i)$ we get: $P(E_j|F_i) = P_j/(1 - P_i\alpha_i)$ for $j \neq i$ and $P(E_i|F_i) = (1 - \alpha_i)P_i/(1 - P_i\alpha_i)$.