Mathematical Theory of Probability(640:477:03)
Fall 2013
Solutions to Assignment 1 [1]

- Chapter 1, problem 10c. We arrange 4 men and 4 women in a row such that no 2 men sit next to each other and no 2 women sit next to each other. Decide the order of the men (4! choices) then decide the order of the women (4! choices), then decide whether to start with a man or a woman (2 choices). Once those choices are made, the order is specified since we must alternate man-woman so there are $2 \times (4!)^2$ ways to do this.
- Chapter 1, problem 10e. We are given 4 married couples and must arrange the 8 people in a line so that each person sits next to their spouse. We count the arrangements as follows. First order the men from left to right (4! choices). For each woman place them to the left or right of their spouse (2 choices) This gives $4!2^4$ arrangements.
- Chapter 1, problem 15. We can break down the choice into three choices: choose the 5 women, then choose the 5 men, then choose the pairing. There are $\binom{10}{5}$ ways to choose the women, $\binom{12}{5}$ ways to choose them men. Given 5 men and 5 women there are 5! ways to pair them off (assign a man to the first woman (5 choices) then assign a man to the second woman (4 choices), etc. Overall the number of choices is $\binom{10}{5}\binom{12}{5}5!$.
- Chapter 2, problem 7. (a) Each person has 2 choices for job type and 3 choices for political affiliation, so there are $6 = 2 \times 3$ choices for the pair. There are 15 people, so overall the size of the sample space is 6^{15}. (b) To count the outcomes with at least one blue collar worker we count outcomes with no blue collar worker and subtract this from the total number of outcomes. For the outcomes with no blue collar, the only choose for each team member is the the political affiliation so there are 3^{15} choices with no blue collar workers. So the number with at least one blue collar worker is $6^{15} - 3^{15}$. (c) If no one is an Independent, then the number of classifications of a given individual is 4 (rather than 6) so the number of outcomes is 4^{15}.
- Chapter 2, problem 14. Let S be the set of subscribers, A be the subset of professionals, B be the subset of married subscribers, and C be the subset of college graduates. Suppose the given data is correct, we derive a contradiction. If we choose a subscriber uniformly at random then Proposition 4.4 tells us that $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$. Using the given data to substitute we get $P(A \cup B \cup C) = .312 + .47 + .525 - .086 - .147 - .042 + .025 = 1.057$. This is impossible since $P(A \cup B \cup C)$ is the probability of an event which is at most 1 (by axiom 1). Therefore we conclude that the given data is faulty.
- Chapter 2, problem 20. Let A be the event that I am dealt a blackjack and B be the event that the dealer is dealt a blackjack. The probability we want is $1 - P(A \cup B)$ which (by Proposition 4.3) is $1 - P(A) - P(B) + P(AB)$. To compute $P(A)$ we note that there are $\binom{52}{2}$ ways to choose 2 cards for me. Of these, there are $64 = 4 \times 16$ ways for the two cards to be blackjack (4 choices for the ace and 16 choices for a 10,Jack,Queen or King.) So $P(A) = 64/\binom{52}{2}$. $P(B)$ is the same. To compute $P(AB)$ we note that there are $\binom{52}{2}$ ways to choose my cards, and then $\binom{50}{2}$ ways to choose the dealer's cards, so the sample space has size $\binom{52}{2}\binom{50}{2}$. Out of these, for us both to have blackjack, I must have an ace (4 choices), and a 10,J,Q,or K (16 choices) and the dealer must have an ace (3 remaining choices) and a 10,J,Q or K (15 remaining choices) for a total of $4 \times 16 \times 3 \times 15$ choices. So $P(AB) = (4 \times 16 \times 3 \times 15)/(\binom{52}{2}\binom{50}{5})$. Thus:

$$1 - P(A \cup B) = 1 - 2 \times \frac{64}{\binom{52}{2}} + \frac{4 \times 3 \times 16 \times 15}{\binom{52}{2}\binom{50}{2}} \approx .9052.$$

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- Chapter 2, problem 21. (a) There are a total of 20 families. For each i between 1 and 5, let N_i be the number of families with i children. Choose a family at random. Let E_i be the event that the chosen family has exactly i children. With the given data:

$$\begin{aligned} P(E_1) &= 4/20 \\ P(E_2) &= 8/20 \\ P(E_3) &= 5/20 \\ P(E_4) &= 2/20 \\ P(E_5) &= 1/20 \end{aligned}$$

  (b) For each i between 1 and 5, let M_i be the number of children that belong to families with exactly i children. Then $M_i = i \times N_i$. For the given data, $M_1 = 4$, $M_2 = 16$, $M_3 = 15$, $M_4 = 8$ and $M_5 = 5$. There are a total of 48 children. Choose a child at random and let F_i be the event that the chosen child is in a family with i children. Then $P(F_i) = |M_i|/48$. For the given data,

$$\begin{aligned} P(F_1) &= 4/48 \\ P(F_2) &= 16/48 \\ P(F_3) &= 15/48 \\ P(F_4) &= 8/48 \\ P(F_5) &= 5/48. \end{aligned}$$

- Chapter 2, problem 25. Let E be the event that the game stops with a roll of 5. Since the game did not stop earlier, no 5 or 7 was rolled before. Let S_n be the event that the game stops after exactly n rolls. The events S_n are disjoint and the sum of their probabilities is 1. Let E_n be the event $E \cap S_n$. The events $E_1, E_2, \ldots,$ are disjoint and there union is E so $p(E) = \sum_{n \geq 1} p(E_n)$. Now E_n is the event that the game ends at step n with a 5 and the probability of that is the probability that each of the first $n-1$ rolls total to neither 5 nor 7 and the last roll sums to 5. There are 36 outcomes of a pair of dice so if we roll the dice n times there are 36^n outcomes. Of the 36 outcomes for 2 dice, 6 of them sum to 7 and 4 of them sum to 5, so 26 outcomes sum to neither 5 nor 7. So the probability of event E_n is $(26^{n-1} \times 4)/36^n = \frac{1}{9}(\frac{26}{36})^{n-1}$. We then have:

$$P(E) = \sum_{n=1}^{\infty} \frac{1}{9}(\frac{26}{36})^{n-1} = \frac{1}{9}\frac{1}{1 - 26/36} = 2/5.$$

- Chapter 2, problem 31. (a) When we choose a player at random from a 3 person team there are 3 outcomes, which we'll call G, F and C. When we choose a player at random from each of 3 teams there are $3^3 = 27$ outcomes. To have a complete team the types of the three chosen players must be different, which happens in $3! = 6$ ways so the probability that we get a complete team is $6/27 = 2/9$. (b) There are three outcomes: GGG, FFF and CCC in which all players are at the same position so the probability this happens is 3/27=1/9.
- Chapter 2, problem 42. If we roll the pair of dice once, there are 35 (out of 36 possible outcomes) ways not to get double 6. If we roll them n times there are 35^n ways (out of 36^n possible outcomes) not to get double 6. Thus the probability of at least one double 6 in n rolls is $1 - (35/36)^n$. The subtracted quantity decreases to 0 as n gets large, so $1 - (35/36)^n$ increases to 1 as n gets large. When n is large enough so that $(35/36)^n \leq 1/2$ the probability is at least 1/2. Using a calculator we see that $(35/36)^{24} \approx .5086$ and $(35/36)^{25} \approx .494$ so we need to roll the dice 25 times for the probability of seeing a double 6 to be at least 1/2.
- Chapter 2, Theoretical exercise 5. We are given a list of events $E_1, E_2, \ldots$ and must define new events $F_1, F_2, \ldots$ in terms of the events $E_1, E_2, \ldots$ so as to satisfy the conditions that

F_i and F_j are disjoint for all $i \neq j$ and for each $n \geq 1$, $\bigcup_{n\geq 1} F_n = \bigcup_{n\geq 1} E_n$. We define the sets $F_1, F_2, \ldots$ as follows $F_1 = E_1$ and for $n \geq 2$, $F_n = E_n - (\bigcup_{h=1}^{n-1} E_h)$.

We claim that the sets $F_1, F_2, \ldots$ defined this way meet the requirements. If F_i and F_j are these sets with $i < j$ any element of F_i is also an element of E_i, but F_j contains no element of E_i so F_i and F_j are disjoint.

We also need to explain why for each $n \geq 1$ the union of $E_1, \ldots, E_n$ is the same as the union of $F_1, \ldots, F_n$. Since $F_i \subseteq E_i$ for each i so we know that $F_1 \cup \cdots \cup F_n$ is a subset of $E_1 \cup \cdots \cup E_n$. We must show also that $E_1 \cup \cdots \cup E_n$ is a subset of $F_1 \cup \cdots \cup F_n$. If x is an element of $E_1 \cup \cdots \cup E_n$ and E_j is the first set in the list of E's that contains x (so $j \leq n$), then by definition of F_j we have $x \in F_j$ so x belongs to the union $F_1 \cup \cdots \cup F_n$.