Intro to Mathematical Reasoning (Math 300)
Supplement 5. Some sample proofs. [1]

This supplement contains some examples of carefully written proofs. In the middle of some of the proofs I include some comments, written as "[COMMENT: ...]". The comments are not part of the proof, but provide some explanation for the thinking in the proof.

Example. Prove: For any three sets A, B, C if $B - C \subseteq A$ then $B - A \subseteq C$.

Proof. Let $\tilde{A}, \tilde{B}, \tilde{C}$ be arbitrary sets. Assume $\tilde{B} - \tilde{C} \subseteq \tilde{A}$. We must show $\tilde{B} - \tilde{A} \subseteq \tilde{C}$.

Let $\tilde{x} \in \tilde{B} - \tilde{A}$ be arbitrary. We must show $\tilde{x} \in \tilde{C}$. We do this by contradiction.

Assume for contradiction that $\tilde{x} \notin \tilde{C}$. We have $\tilde{x} \in \tilde{B}$ and $\tilde{x} \notin \tilde{A}$. Since $\tilde{x} \in \tilde{B}$ and $\tilde{x} \notin \tilde{C}$ we have $\tilde{x} \in \tilde{B} - \tilde{C}$. Since $\tilde{B} - \tilde{C} \subseteq \tilde{A}$ we have $\tilde{x} \in \tilde{A}$. But this contradicts $\tilde{x} \notin \tilde{A}$. Therefore we conclude that our assumption $\tilde{x} \notin \tilde{C}$ must be wrong and so $\tilde{x} \in \tilde{C}$ as required. □

Example. In this problem, for each natural number n, A_n denotes the set $(2 - 1/n, 4 - 1/n]$ and B_n denotes the set $[1 + 1/n, 4 - 2/n)$. Prove $\bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}} B_n$.

Proof. We must prove that for all $x \in \bigcup_{n\in\mathbb{N}} A_n$, we have $x \in \bigcup_{n\in\mathbb{N}} B_n$.

Let $\tilde{x}$ be an arbitrary member of $\bigcup_{n\in\mathbb{N}} A_n$. We must show that $x \in \bigcup_{n\in\mathbb{N}} B_n$. By definition of union, this means that we must show that there exists an $n \in \mathbb{N}$ such that $\tilde{x} \in B_n$. [COMMENT: This is our new goal.]

Since $\tilde{x} \in \bigcup_{n\in\mathbb{N}} A_n$, there is a natural number we'll call $\hat{k}$ such that $\tilde{x} \in A_{\hat{k}}$. By the definition of $A_{\hat{k}}$, this means $2 - 1/\hat{k} < \tilde{x}$ and $4 - 1/\hat{k} \geq \tilde{x}$.

[COMMENT: Returning to our goal, we need to provide instructions for choosing a natural number $\hat{n}$ so that $\tilde{x} \in B_{\hat{n}}$. Our instructions will depend on $\hat{k}$. The instructions given below may seem mysterious: why did I choose $\hat{n}$ the way I did? The short answer is: to make things work out. To figure out the instructions for $\hat{n}$ requires some careful thinking, including considering examples.]
We divide into two cases, depending on whether $\tilde{x} \leq 2$ or not.

Case 1. Assume $\tilde{x} \leq 2$. Since $\tilde{x} > 2 - 1/k$ and $k \geq 1$, we have $\tilde{x} > 1$ and so $\tilde{x} - 1 > 0$. Let $\hat{n}$ be any integer larger than $1/(\tilde{x} - 1)$. We claim $\tilde{x} \in B_{\hat{n}}$. So we need (i) $\tilde{x} \leq 4 - 2/\hat{n}$ and (ii) $\tilde{x} \geq 1 + 1/\hat{n}$. For (i), by the case assumption, we have $\tilde{x} \leq 2$ and since $\hat{n} \geq 1$ we have $2 \leq 4 - 2/\hat{n}$ so $\tilde{x} \leq 4 - 2/\hat{n}$, as needed. For inequality (ii), we have $1 + 1/\hat{n} \leq 1 + 1/(1/(\tilde{x} - 1)) = 1 + \tilde{x} - 1 = \tilde{x}$ as required to prove this case. So $\tilde{x} \in \bigcup_{n\in\mathbb{N}} B_n$ in this case.

Case 2. Assume $\tilde{x} > 2$. In this case we choose $\hat{n} = 3\hat{k}$. We claim $\hat{n}$ is an integer and $\tilde{x} \in B_{\hat{n}}$. Since $\hat{k}$ is an integer, so is $\hat{n}$. To show $\tilde{x} \in B_{\hat{n}}$, we need (i) $\tilde{x} \leq 4 - 2/\hat{n}$ and (ii) $\tilde{x} \geq 1 + 1/\hat{n}$. For inequality (ii), from the case assumption we have $\tilde{x} > 2$, and $2 \geq 1 + 1/\hat{n}$ since $\hat{n} \geq 1$, so $\tilde{x} \geq 1 + 1/\hat{n}$. For inequality (i), we have $\tilde{x} \leq 4 - 1/\hat{k}$ since $\tilde{x} \in A_{\hat{k}}$ and we have $4 - 2/\hat{n} = 4 - \frac{2}{3\hat{k}} \geq 4 - 1/\hat{k}$ since $\hat{k} > 0$. Therefore $\tilde{x} \leq 4 - 2/\hat{n}$, as needed to finish the proof of this case. So $\tilde{x} \in \bigcup_{n\in\mathbb{N}} B_n$ in this case.

Since $\tilde{x}$ was an arbitrary member of $\bigcup_{A\in\mathcal{A}} A$, we conclude that $\bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}} B_n$. □

Remark. If one considers the two unions that appear in the statement to be proved, one can see that each of these unions is equal to the set $(1, 4)$. So an alternative (and clearer) proof would be to prove separately that $\bigcup_{n\in\mathbb{N}} A_n = (1, 4)$ and that $\bigcup_{n\in\mathbb{N}} B_n = (1, 4)$. This is left as an exercise.

Example.
Theorem. Prove that for any two intervals I, J, if $I - J$ is not an interval then $J \subseteq I$.

Proof. Let $\tilde{I}$ and $\tilde{J}$ be arbitrary intervals. Assume $\tilde{I} - \tilde{J}$ is not an interval. We must show $\tilde{J} \subseteq \tilde{I}$. So let $\tilde{x}$ be an arbitrary member of $\tilde{J}$. We must show $\tilde{x} \in \tilde{I}$.

Since $\tilde{I} - \tilde{J}$ is not an interval, by the definition of interval, there must exists real numbers that we'll call $\hat{a}, \hat{b}, \hat{c}$ that satisfy $\hat{a} < \hat{b} < \hat{c}$ and $\hat{a}, \hat{c} \in \tilde{I} - \tilde{J}$ but $\hat{b} \notin \tilde{I} - \tilde{J}$.

We claim that $\hat{b}$ is a member of both $\tilde{I}$ and $\tilde{J}$. We have $\hat{b} \in \tilde{I}$ since $\hat{a} < \hat{b} < \hat{c}$ and $\hat{a}$ and $\hat{c}$ both belong to the interval $\tilde{I}$. Since $\hat{b} \notin \tilde{I} - \tilde{J}$ and $\hat{b} \in \tilde{I}$, it must be that $\hat{b} \in \tilde{J}$.

We will show that $\hat{a} < \tilde{x} < \hat{c}$ which will allow us to conclude $\tilde{x} \in \tilde{I}$.

We first use proof by contradiction to show that $\tilde{x} > \hat{a}$. Assume for contradiction that $\tilde{x} \leq \hat{a}$. We can't have $\tilde{x} = \hat{a}$ since $\hat{a} \notin \tilde{J}$ but $\tilde{x} \in \tilde{J}$ so $\tilde{x} < \hat{a}$. Since also $\hat{a} < \hat{b}$ and $\tilde{x}, \hat{b} \in \tilde{J}$ and $\tilde{J}$ is an interval, we would have $\tilde{a} \in \tilde{J}$, contradicting that $\tilde{b} \in \tilde{I} - \tilde{J}$. Therefore, $\tilde{x} > \hat{a}$.

Next we use proof by contradiction to show that $\tilde{x} < \hat{c}$. Assume for contradiction that $\tilde{x} \geq \hat{c}$. We can't have $\tilde{x} = \hat{c}$ since $\hat{c} \notin \tilde{J}$ but $\tilde{x} \in \tilde{J}$ so $\tilde{x} > \hat{c}$. Since also $\hat{c} > \hat{b}$ and $\tilde{x}, \hat{b} \in \tilde{J}$ and $\tilde{J}$ is an interval, we would have $\tilde{c} \in \tilde{J}$, contradicting that $\tilde{b} \in \tilde{I} - \tilde{J}$. Therefore, $\tilde{x} < \hat{c}$.

We therefore have $\hat{a} < \tilde{x} < \hat{c}$ and $\hat{a}, \hat{c}$ are in the interval $\tilde{I}$. So $\tilde{x} \in \tilde{I}$. Since $\tilde{x}$ was an arbitrary member of $\tilde{J}$, we conclude $\tilde{J} \subseteq \tilde{I}$.

We therefore conclude that for any two intervals I, J if $I - J$ is not an interval then $J \subseteq I$. $\square$.

Example. Recall that an integer is square-free if it has no divisor larger than 1 that is a square.
Theorem. For all integers n, n is not square-free if and only if there are integers a, b, c such that $n = ab$ and $c|a$ and $c|b$ and $c > 1$.

(Before reading the proof, do some examples to see that you understand this.)
Proof. Let $\tilde{n}$ be an arbitrary integer. We must show two things:

(1) If $\tilde{n}$ is not square-free then there are integers a, b, c such that $\tilde{n} = ab$ and $c|a$ and $c|b$ and $c > 1$.

(2) If there are integers a, b, c such that $\tilde{n} = ab$ and $c|a$ and $c|b$ then $\tilde{n}$ is not square-free.

Proof of (1). Assume $\tilde{n}$ is not square-free. We must show there are integers a, b, c with $\tilde{n} = ab$ and $c|a$ and $c|b$ and $c > 1$.

By definition, there is a square we'll call $\hat{s}$ such that $\hat{s} > 1$ and $\hat{s}|n$. By definition of square, there is a natural number we'll call $\hat{c}$ such that $\hat{c}^2 = \hat{s}$. If $\hat{c}$ is equal to 1 then $\hat{s}$ is also equal to 1, which is impossible so $\hat{c} > 1$. Since $\hat{s}|\tilde{n}$, there is an integer we'll call $\hat{t}$ such that $\hat{s}\hat{t} = \tilde{n}$. Let $\hat{a} = \hat{c}$ and $\hat{b} = \hat{c}\hat{t}$. Then $\tilde{n} = \hat{a}\hat{b}$ and $\hat{c}|\hat{a}$ and $\hat{c}|\hat{b}$ and $\hat{c} > 1$, as required to complete the proof of this part.

Proof of (2). Assume there are integers which we'll call $\hat{a}, \hat{b}, \hat{c}$ with $\tilde{n} = \hat{a}\hat{b}$ and $\hat{c}|\hat{a}$ and $\hat{c}|\hat{b}$ and $\hat{c} > 1$. We must show that $\tilde{n}$ is not square free. Since $\hat{c}|\hat{a}$ there is an integer we'll call $\hat{j}$ such that $\hat{c}\hat{j} = \hat{a}$. Similarly there is an integer we'll call $\hat{k}$ such that $\hat{c}\hat{k} = \hat{b}$. Therefore $\tilde{n} = \hat{a}\hat{b} = \hat{c}^2\hat{j}\hat{k}$ and so $\hat{c}^2$ is a square that is greater than 1 and is a divisor of $\tilde{n}$. So $\tilde{n}$ is not square-free, as required to prove (2).

□