Intro to Mathematical Reasoning (Math 300)

Supplement 2. **Proving universal and existential propositions** [1]

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1 Theorems and conjectures

As mathematicians and students of mathematics, it is our job to separate those mathematical propositions that are true from those that are false. The method that mathematicians use to verify that a proposition is true is called *deductive proof*. Roughly speaking, a deductive proof is a *step by step* argument that uses *known facts* and applies *valid rules of deduction* to build to a desired conclusion.

A mathematical proposition that has been proved to be true is called a *Theorem*. There are other words that are sometimes used for proved mathematical propositions:

- The word *corollary* is generally used to refer to a a proved mathematical proposition that is deduced as an easy consequence of a previously proved theorem.
- The word *lemma* refers to a proved mathematical proposition that is mainly of interest because it is a step in the proof of a theorem.

A proposition that a mathematician *believes* to be true based on some good evidence, but for which no proof is known, is called a *conjecture*. For example, one of the most famous conjectures in mathematics is *Goldbach's conjecture*:

Every even number greater than 2 can be expressed as the sum of two primes.

It has been verified by computer that every even number between 4 and 4×10^{14} can be expressed as the sum of two primes. However, no one knows whether this is true of all even numbers.

2 Why do we need rules for mathematical proof?

As you will learn, to be valid, a mathematical proof must be constructed in accordance with certain rules. Early in the course, most beginning students will find that the proofs they submit are often rejected as *faulty* because they fail to follow these rules. Before discussing what these rules are, it is important to understand why there are rules for mathematical proof.

In any activity, certain occurences are recognized as disasters. In rock climbing, falling off a rock face from 100 feet without being secured to the rock by a rope is a disaster. In mathematics, the biggest disaster that can happen is *to declare that a mathematical proposition has been proved to be true*, when actually *the mathematical proposition is false*. It is a mathematical disaster because it undermines our search for mathematical truth, and it is a personal disaster because it can be quite embarassing. In some circumstances, it can also be disastrous on a practical level: if a mistaken theorem is used to plan the orbit of a satellite, the satellite may crash or be lost in space.

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The rules for mathematical proof are designed to prevent this disaster from occuring; if you follow the rules *it is impossible to accidentally give a proof of a false proposition.*

A proposed proof that does not follow the rules for mathematical proof is called a *faulty proof.* A proof can be faulty even if the proposition it is trying to prove is true. Such proofs are not permitted because they use arguments that could be used to prove false propositions.

3 What propositions need to be proved?

In this course, you will often be asked to prove some very basic facts such as: "The sum of any two even numbers is even." This is something you've probably known for years and it may seem pointless to prove such obvious propositions.

There are several reasons for learning to prove such propositions:

- Writing mathematical proofs is a difficult skill. It makes sense to start by practicing on easy examples. Later, when we prove more difficult theorems, the proofs will be more involved and require more sophisticated ideas. In order to prepare for more difficult proofs, you need to master easy ones first.
- While you may think that you know that the sum of any two even numbers is even, how can you be sure? Sometimes propositions that seem obvious are not even true. No one has (or can) ever check every pair of even numbers because there are infinitely many of them. A mathematical proof of this fact will provide a convincing reason that guarantees that this is really true.
- One of our goals here is to see how mathematics is built up from simple facts. By proving that the sum of any two even numbers is even, we see how this simple fact is built up from even more basic facts.

4 Where do we start?

Proofs use known facts to establish new ones. Once we prove a theorem, that theorem joins our set of known facts, and we can use it to prove later theorems. But how do we get started? What facts can we use the first time we prove a theorem?

In order to start proving theorems, we have to agree on a starting point: a collection of known concepts and facts that we are allowed to build with. Mathematicians refer to the starting facts as axioms. You can think of the axioms as the initial set of tools that you are allowed to use.

There is no one choice of axioms that is the "right" choice. The known facts that you use in your proofs depends on your audience. If you are writing a proof to be read by research mathematicians that you can assume that they already know many advanced facts, and you don't have to prove them. But if you are writing a proof for a beginning student, then you should only use facts that are well known and well understood by them.

At this point, it is too tedious to give a precise list of what facts can and can't be assumed when doing your proofs. The best we can do is give some guidelines. Here are facts you may use in your proofs about real numbers (unless otherwise instructed).

- *Arithmetic facts.* Arithmetic facts involving exact calculations with specific integers and rational numbers that you could do by hand such as "$\frac{2}{3} + \frac{1}{5} = \frac{13}{15}$", "$453 - 28 = 425$", "$3^5 = 243$", and "52 is not a multiple of 7".

- *Simple algebraic manipulation.* Any computation involving addition, subtraction, multiplication and division that can be justified by applying the commutative, distributive and associative laws of arithmetic. For example, you may say that the proposition "For all real numbers a,b, $(a + b) \times (a + b) = a^2 + 2ab + b^2$." is true by simple algebraic manipulation because the left hand side can be transformed into the right hand side by applying the commutative, distributive and associative laws. Facts involving division may require some care to ensure that you don't divide by 0.

- Basic facts about the number 0, such as "For any two real numbers, if their product is 0 then one of them must be 0."

- Basic facts about inequalities such as: "for all real numbers a, b, c, if $b \geq c$ then $a+b \geq a+c$." Again these facts require care in applying them. For example, it is not true that "for all real numbers a, b, c, if $b \geq c$ then $ab \geq ac$" because there are counterexamples. It is true that "for all real numbers a, b, c, if $b \geq c$ and $a \geq 0$ then $ab \geq ac$."

There is a basic principle that is used all the time, that is so obvious that it is often not mentioned. This is the *substitution principle*: Whenever y and z are numbers (or other mathematical objects) and it is known that $y = z$, if we are given a true proposition about y then we may substitute z for y in the proposition to get a new true proposition. When we say that something is true "by substitution" we mean that we are using this principle.

Here are some examples of facts that, at this point in the course, *can not* be taken as known facts for your proofs:

- Facts involving square roots, cube roots and higher roots of real numbers. For example, you may not assume the fact that every nonnegative real number has a square root.

- When proving facts about the set of integers, unless otherwise stated, you may not assume facts about division. Also you may not assume facts involving prime numbers, divisibility, etc. (such as the fact that every positive integer greater than 1 can be written as the product of prime numbers).

- Facts involving calculus.

At different points in the course, the facts we will take as known facts may change. For example, when we are proving propositions about the integers (rather than about the real numbers), involving concepts like prime numbers, divisor, etc. then we will consider that basic facts about addition, subtraction and multiplication are known, but not basic facts about division.

Important: If you have doubts about whether you are allowed to use a particular known fact in a proof, ask your instructor!

5 Formal and informal proof

There are different styles of mathematical proof, from strictly formal proofs to informal. In *strictly formal proofs*, the rules are very clear and precise. Formal proofs are written symbolically rather than in English. These proofs are the least prone to error but they are also very hard to work with because the symbolic language they use is very hard to read. Also, strictly formal proofs, even of very simple propositions, tend to be extremely long. We will not learn how to do formal proofs in this course.

Because strictly formal proofs are so tedious, mathematicians rarely write them. Instead, they write informal proofs, which are written in some "natural" language (which, for us, is English) augmented with mathematical terminology and symbols. *These are the kinds of proofs you will learn to do in this course.* Informal proofs written in English should consist of complete sentences and follow the the usual grammatical rules of English. In addition, there are the rules for making proper mathematical deductions. Unfortunately, the rules for informal proof are harder to specify than for formal proofs. In this supplement and the following ones these rules will be explained. Writing informal proofs is learned largely from examples, and many examples will be given.

6 Proving Existential Propositions

A proposition of the form "There is an x satisfying $P(x)$" simply says that there is at least one choice of an object x that makes $P(x)$ true. So to prove this, you need only produce one example of an object that makes $P(x)$ true, and then explain why it makes $P(x)$ true.

Disproving a universal proposition is the same as *proving* an existential proposition. To disprove

> For all x, $Q(x)$.

means to prove the negation:

> It is not the case that for all x, $Q(x)$.

and this negation is equivalent to proving the existential proposition:

> There is an x satisfying $\sim Q(x)$".

We've already seen examples of proving existential propositions, since in supplement 1 and homework 1 we disproved some universal propositions. Here's one more example:

> **Theorem.** There exist three positive integers with the property that no one of the integers is a multiple of another one, but the product of any two is a multiple of the third.

Proof. Select the integers 6, 10 and 15.
We have that:

- 6 is not a multiple of 10 or of 15
- 10 is not a multiple of 6 or of 15 and

- 15 is not a multiple of 10 or 6.

Furthermore,

- $6 \times 10 = 4 \times 15$ so 6×10 is a multiple of 15.
- $6 \times 15 = 9 \times 10$ and so 6×15 is a multiple of 10.
- $10 \times 15 = 25 \times 6$ and so 10×15 is a multiple of 6.

□

Exercise. Find another example that satisfies the requirements of the Theorem.

The symbol "□" is a common way to indicate that your proof is ended. An alternative is to use the Latin abbreviation "QED" for "quod erat demonstrandum" which means "that which was to be proven".

7 Proving Universal Propositions

We have just seen that to prove an existential proposition involves producing and verifying one example of the associated predicate. How do we go about proving a universal proposition? A universal proposition has the form "For all x that satisfy the hypothesis $H(x)$, the conclusion $C(x)$ must also be true." As we've said before, a single universal proposition summarizes many (often infinitely many) different propositions, one for each value of the variable x that makes $H(x)$ true. Somehow, we must prove that $C(x)$ holds, knowing that $H(x)$ is true but not knowing what x is. How can we show that something is true for all possible values of x?

There is one situation where this is easy. If we can show that there are only a small finite number of choices for x that make $H(x)$ true, then we can just check each one to see if $C(x)$ is true. For example here's a proof of:

> Every positive integer less than 5 is a root of the polynomial p defined by $p(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$.

Proof. The only positive integers less than 5 are 1,2,3 and 4. By elementary arithmetic, we check $p(1) = 1-10+35-50+24 = 0$, $p(2) = 16-80+140-100+24 = 0$, $p(3) = 81-270+315-150+24 = 0$, and $p(4) = 256 - 640 + 560 - 200 + 24 = 0$, so $1, 2, 3$ and 4 are all roots of p. Therefore every positive integer less than 5 is a root of $x^4 - 10x^3 + 35x^2 - 50x + 24$. □

However, for most interesting universal propositions, the number of x we have to consider is infinite, and then it is not possible to try all possibilities.

We could select a few values for x that that satisfy $H(x)$ and check that they satisfy $C(x)$. This is a useful way to gain understanding about the universal proposition, but it does not prove the statement.

So now we come to the most important method for proving universal propositions. The basic method is used by almost all mathematicians in their proofs, but there are different ways to formulate it. Also, the method does not have a standard name. In these notes, we call the method the *hypothetical object method*, but people (including mathematicians) outside the course will probably not be familiar with this name.

The basic idea of the hypothetical object method is this. We want to prove that every object that satisfies hypothesis H also satisfies the conclusion C. We start by imagining that someone else has chosen an arbitrary object that satisfies the hypothesis, but has not told us which object was selected. This object is called the *hypothetical object*. We then use the fact that the object satisfies H to build a *chain of deductions* that ends with showing that the object satisfies C. Each step of the proof must be valid regardless of what the hypothetical object is.

This is all very vague. How to we build this "chain of deductions"? What is a "chain of deductions" anyway? We will need to see some examples.

Example. We will use the hypothetical object method to prove:

Theorem. The product of any two real numbers is at most the square of their average.

It is a good habit, before proving any universal proposition, to try some examples to make sure you understand what the proposition is saying. So we select some objects satisfying the hypothesis, say the numbers 7 and 12. Then we test the conclusion. Their product is 84. Their average is $(7+12)/2$ which is 9.5 and $9.5^2 = 90.25$ which is, indeed, bigger than 84. So the example does illustrate the proposition, but keep in mind that this example is not a proof!

We now present a proof of the Theorem, followed by some explanations and comments about the steps in the proof.

Proof. Let $\tilde{r}$ and $\tilde{s}$ be arbitrary real numbers. [*Comment: $\tilde{r}$ and $\tilde{s}$ are our "hypothetical objects".*] Our goal is to show:

$$\tilde{r}\tilde{s} \leq (\frac{\tilde{r}+\tilde{s}}{2})^2.$$

To do this we will show:

$$(\frac{\tilde{r}+\tilde{s}}{2})^2 - \tilde{r}\tilde{s} \geq 0.$$

By definition of "squared", $(\frac{\tilde{r}+\tilde{s}}{2})^2 = (\frac{\tilde{r}+\tilde{s}}{2}) \times (\frac{\tilde{r}+\tilde{s}}{2})$. By algebraic manipulation,

$$(\frac{\tilde{r}+\tilde{s}}{2}) \times (\frac{\tilde{r}+\tilde{s}}{2}) = \frac{1}{4}(\tilde{r}^2 + 2\tilde{r}\tilde{s} + \tilde{s}^2).$$

Subtracting $\tilde{r}\tilde{s}$ from both sides we obtain:

$$(\frac{\tilde{r}+\tilde{s}}{2}) \times (\frac{\tilde{r}+\tilde{s}}{2}) - \tilde{r}\tilde{s} = \frac{1}{4}(\tilde{r}^2 + 2\tilde{r}\tilde{s} + \tilde{s}^2) - \tilde{r}\tilde{s}.$$

By algebraic manipulation, the right hand side of this equation is equal to $(\frac{\tilde{r}-\tilde{s}}{2})^2$, and so:

$$(\frac{\tilde{r}+\tilde{s}}{2})^2 - \tilde{r}\tilde{s} = (\frac{\tilde{r}-\tilde{s}}{2})^2.$$

Since the square of any real number is greater than or equal to 0, this last quantity is greater than or equal to 0 and so:

$$(\frac{\tilde{r}+\tilde{s}}{2})^2 - \tilde{r}\tilde{s} \geq 0.$$

Adding $\tilde{r}\tilde{s}$ to both sides, we conclude that $(\frac{\tilde{r}+\tilde{s}}{2})^2 \geq \tilde{r}\tilde{s}$.

Since $\tilde{r}$ and $\tilde{s}$ were arbitrary real numbers, we conclude that for all two real numbers, their product is at most the square of their average. □

Let's discuss this proof in detail.

1. The statement to be proved is a universal proposition about any choice of two real numbers, so we use the hypothetical object method. The first sentence is

   > Let $\tilde{r}$ and $\tilde{s}$ be arbitrary real numbers.

   Students are sometimes confused by the words "Let" and "arbitrary" here. This particular choice of words is a mathematician's way of saying:

   > I am going to use the hypothetical object method to prove this proposition. In this proof, $\tilde{r}$ and $\tilde{s}$ represent two unknown real numbers, which are the hypothetical objects needed for the proof.

2. We then say clearly what we need to show and our strategy for getting to the goal:

   > $\tilde{r}\tilde{s} \leq (\frac{\tilde{r}+\tilde{s}}{2})^2$. To do this, we will show that $(\frac{\tilde{r}+\tilde{s}}{2})^2 - \tilde{r}\tilde{s} \geq 0$.

3. So far, what we have done can be thought of us "setting up" the proof. Next comes the main part of the proof, which proceeds step by step to demonstrate that, whatever two numbers $\tilde{r}$ and $\tilde{s}$ were chosen, the desired conclusion holds. Each step consists of building on what has been established, and using known facts from algebra to draw a new conclusion until we reach our goal. Each step of the proof is justified by stating what general facts we are using.

4. The final paragraph summarizes by saying that since we accomplished what is needed for the hypothetical object method, the universal statement is proved.

A remark on the names of objects in proofs. In nearly any proof that uses the hypothetical object method, one assigns variable names to the hypothetical objects. Most mathematicians use a single letter name. In these notes, as an aid to learning the method, I use a letter with a "$\sim$" (tilde) over it. Students are encouraged to do the same. (Your instructor may require it.) If you use tildes over hypothetical objects, do not use them over letters that do not represent hypothetical objects. (See supplement 3 for more details on the different use of letters in proofs.)

Here's another example of the *hypothetical object method.*

Example. We will use the hypothetical object method to prove:

Theorem. For all real numbers x and y, $|x+y| \leq |x| + |y|$.

Again before proving this, we illustrate it with an example: If we choose $x = 7$, $y = -4$ then $|7 + (-4)| = 3$ which is less than $|7| + |-4| = 11$.

We present the proof below followed by a discussion of the proof.

Proof. Let $\tilde{x}$ and $\tilde{y}$ be arbitrary real numbers.

We must show that $|\tilde{x} + \tilde{y}| \leq |\tilde{x}| + |\tilde{y}|$.

We know that either $\tilde{x} + \tilde{y} \geq 0$ or $\tilde{x} + \tilde{y} < 0$. So we analyze these two possibilities separately.

Case 1. Assume $\tilde{x} + \tilde{y} \geq 0$. Then by the definition of absolute value, $|\tilde{x} + \tilde{y}| = \tilde{x} + \tilde{y}$. So it is enough to show that $\tilde{x} + \tilde{y} \leq |\tilde{x}| + |\tilde{y}|$. Now, since for any real number z, $|z| \geq z$ we conclude that $\tilde{x} \leq |\tilde{x}|$ and $\tilde{y} \leq |\tilde{y}|$. Adding these two inequalities together, we get:

$$\tilde{x} + \tilde{y} \leq |\tilde{x}| + |\tilde{y}|.$$

Recall that in the case being considered, $|\tilde{x} + \tilde{y}| = \tilde{x} + \tilde{y}$, so we get the desired conclusion:

$$|\tilde{x} + \tilde{y}| \leq |\tilde{x}| + |\tilde{y}|.$$

Case 2. Assume $\tilde{x} + \tilde{y} < 0$. Then by the definition of absolute value, $|\tilde{x} + \tilde{y}| = -(\tilde{x} + \tilde{y}) = (-\tilde{x}) + (-\tilde{y})$. So it is enough to show that $(-\tilde{x}) + (-\tilde{y}) \leq |\tilde{x}| + |\tilde{y}|$. Now, since for any real number z, $|z| \geq -z$, we conclude that $-\tilde{y} \leq |\tilde{y}|$ and $-\tilde{x} \leq |\tilde{x}|$. Adding these two inequalities together, we get:

$$(-\tilde{x}) + (-\tilde{y}) \leq |\tilde{x}| + |\tilde{y}|.$$

Recall that in the case being considered, $|\tilde{x}+\tilde{y}| = (-\tilde{y})+(-\tilde{x})$, so we get the desired conclusion:

$$|\tilde{x} + \tilde{y}| \leq |\tilde{x}| + |\tilde{y}|.$$

Since in both cases, we have shown that $|\tilde{x} + \tilde{y}| \leq |\tilde{x}| + |\tilde{y}|$ and the two cases together cover all possibilities, we conclude that for every choice of two real numbers x and y, $|x + y| \leq |x| + |y|$.

□

Let's discuss this proof in detail.

1. As with any proof using the hypothetical object method we start by introducing the objects that satisfy the hypothesis. Notice that the names we use are similar to, but not the same, as the variables appearing in the statement of the Theorem. There is a relationship between $\tilde{x}$ and x: $\tilde{x}$ stands for the unknown value selected for x, but $\tilde{x}$ is not the same as x and you should be careful not to confuse the two. Similarly $\tilde{y}$ is the unknown value selected for y, but is not the same as y.

2. In this proof, it is convenient to split the work of the proof into two parts, using a technique called *Proof by cases.* This technique will be discussed in more detail in a later supplement. Here we looked at whether $\tilde{x} + \tilde{y} \geq 0$ or $\tilde{x} + \tilde{y} < 0$. We don't know which of these is true (since we don't know what $\tilde{x}$ and $\tilde{y}$ are) but we know that one of them must be true. So in case 1, we give one argument if the first possibility is true and in case 2, we give a different argument that works if the second possibility is true. Since one of these possibilities must be true, this is enough to complete the proof.

3. The proofs in the two cases are not the same, but they are very similar.

Example. *Uniqueness theorems.* Consider the following example:

> **Proposition.** There is a unique real number that is a root of the equation $x^3 + 37 = 0$.

This proposition means that two things are true:

1. There is *at least one* real number that is a root of $x^3 + 37 = 0$, and
2. There is *at most one* real number that is a root of $x^3 + 37 = 0$.

Proving this proposition requires proving both of these things, which normally must be proved separately.

The first part is an existential proposition. To prove this, we would use the usual method by which we prove existential propositions. However, by the rules we follow the following *is not an acceptable proof*:

> Let x be the cube root of -37. Then $x^3 = -37$ so $x^3 + 37 = 0$.

Why not? Because the first step assumes that there is a cube root of -37. While it is true that there is a cube root of -37, we said earlier that in this course the basic facts about square roots, cube roots and higher roots are not assumed to be known. So for this proof, we would have to prove that there is a cube root of -37. This is not so easy and requires use of the completeness axiom introduced at the end of this supplement and presented in Section 7.1. We can also prove that there is a cube root of -37 using the "Intermediate value theorem" from calculus, but as we stated earlier, facts from calculus may not be used without proof. So we are not ready to prove the first part.

While we can't yet prove the first part, we can still try to prove the second part, which is a separate proposition. This part says: "There is at most one solution to the equation $x^3 + 37 = 0$"; in other words, this equation either has no solutions or it has precisely one solution.

Is this proposition an existential proposition, a universal proposition or neither? Since the proposition starts with "There is" we might think that it is existential. But an existential proposition is one that can be written in the form "$\exists x$ such that $P(x)$." There is no proposition equivalent to the given proposition that has this form. On the other hand, we can reword the proposition as "There do not exist numbers x and y satisfying $x^3+37 = 0$ and $y^3+37 = 0$ such that $x \neq y$.", which is *the negation of an existential proposition* and is therefore equivalent to a universal proposition. One way to write this universal proposition is:

> For all real numbers x, y, if $x^3 + 37 = 0$ and $y^3 + 37 = 0$ then $x = y$.

Here is a proof using the hypothetical object method.

Proof. Let $\tilde{x}$ and $\tilde{y}$ be real numbers satisfying $\tilde{x}^3 + 37 = 0$ and $\tilde{y}^3 + 37 = 0$. We will show that $\tilde{x} = \tilde{y}$. By the hypothesis we have $\tilde{x}^3 + 37 = \tilde{y}^3 + 37$ so $\tilde{x}^3 = \tilde{y}^3$. Therefore $\tilde{x}^3 - \tilde{y}^3 = 0$. To finish the proof, we will show that $\tilde{x} = \tilde{y}$.

By simple algebraic manipulation, $\tilde{x}^3 - \tilde{y}^3 = (\tilde{x} - \tilde{y})(\tilde{x}^2 + \tilde{x}\tilde{y} + \tilde{y}^2)$. Since $\tilde{x}^3 - \tilde{y}^3 = 0$, we have $(\tilde{x} - \tilde{y})(\tilde{x}^2 + \tilde{x}\tilde{y} + \tilde{y}^2) = 0$. We focus on the second factor, and consider separately the two possibilities that it is not 0, and that it is 0. We will now show that in either case, $\tilde{x} = \tilde{y}$.

Case 1. Assume that $\tilde{x}^2 + \tilde{x}\tilde{y} + \tilde{y}^2 \neq 0$. Then since the product of this with $\tilde{x} - \tilde{y}$ is 0, it must be that $\tilde{x} - \tilde{y} = 0$, so $\tilde{x} = \tilde{y}$.

Case 2. Assume that $\tilde{x}^2 + \tilde{x}\tilde{y} + \tilde{y}^2 = 0$. Adding $\tilde{x}\tilde{y}$ to both sides, we have $(\tilde{x} + \tilde{y})^2 = \tilde{x}\tilde{y}$ and since the first quantity is the square of a real number, we have $\tilde{x}\tilde{y} \geq 0$. But then each of the three numbers $\tilde{x}^2$, $\tilde{x}\tilde{y}$, $\tilde{y}^2$ is nonnegative, while their sum is 0, which means that they must all be 0. Since $\tilde{x}^2 = \tilde{y}^2 = 0$ we have $\tilde{x} = \tilde{y} = 0$.

Thus we have shown that $\tilde{x} = \tilde{y}$ in both cases. We therefore conclude that for all real numbers x, y, $x^3 + 37 = 0$ and $y^3 + 37 = 0$ implies $x = y$. □

Exercise. Using the ideas of the last proof, prove the proposition that for all real numbers x, y, if $y > x$ then $y^3 > x^3$. (Notice that this proposition implies that every real number has at most one cube root.)

8 Proving Universal-Existential Propositions

Recall that a universal proposition whose conclusion is an existential predicate is called a "universal-existential" proposition. The form of such a proposition is:

> For all x satisfying $H(x)$, there exists a y such that x and y together satisfy $Q(x, y)$.

In a universal-existential proposition the variable (or variables) that are universally quantified can be thought of as the *input variable* (or variables) and the variable (or variables) that are existentially quantified can be thought of as the *output variables.* The proposition above can be thought of as:

> If you give me any input x that satisfies $H(x)$, I can find an output y such that $Q(x, y)$ is satisfied.

As we have discussed, it is very important to remember that the order of the quantifiers matters. There is a big difference between a proposition of the above form, and an *existential-universal* proposition which has the form: "There exists y such that for all x satisfying $H(x)$, x and y together satisfy $Q(x, y)$."

Since a universal-existential proposition is a kind of universal proposition, we use the hypothetical object method to prove it. Let's see an example of a simple proof of such a proposition.

Example. Prove: For all real numbers x, if $x < 2$ then there is a real number z such that $x < z$ and $z < 2$.

Here the hypothesis is "x is a real number satisfying $x < 2$." The conclusion is "There is a real number z such that $x < z$ and $z < 2$."

Intuitively, the proposition says that if you give me a number x that is less than 2, I can find a bigger number that is still less than 2. Like many of the things we will prove in this course, this result seems obvious, but since we are learning to write proofs, "seems obvious" is not enough.

Before actually writing the proof, let's plan out how we will do the proof. Since this is a universal proposition, we will try to prove this by the hypothetical object method. So let $\tilde{x}$ be an arbitrary real number less than 2.

Now we want to show "There is a real number z such that $\tilde{x} < z$ and $z < 2$". This is an existential predicate that depends on $\tilde{x}$. As with the proof of an existential proposition we will try to do this by giving instructions for choosing a value for z and then verifying that if this value is chosen according to the instructions, then the conclusion holds. The difficulty here is that we don't know what $\tilde{x}$ is. We want to pick a value for z to be a *little bit bigger* than $\tilde{x}$. But it won't work to say: Choose the value for z to be $\tilde{x} + .0000001$. (Why not?)

What we need to do is to give instructions for choosing the value that depends on $\tilde{x}$. If we take the average of $\tilde{x}$ and 2, $(\tilde{x}+2)/2$, we would expect to get a number between them. Now that we think we know how to give instructions for picking a value for z, we can try to write a proof.

Proof. Let $\tilde{x}$ be an arbitrary real number satisfying $\tilde{x} < 2$. We will show that there exists a real number z such that $\tilde{x} < z$ and $z < 2$. Let $\hat{z} = (\tilde{x}+2)/2$. We show that $\hat{z}$ satisfies $\tilde{x} \leq \hat{z}$ and $\hat{z} < 2$.

Since $\tilde{x} < 2$, we can add $\tilde{x}$ to both sides to get $2\tilde{x} < \tilde{x}+2$. We then multiply both sides by $1/2$ to get $\tilde{x} < (\tilde{x}+2)/2$. Since the expression on the right is $\hat{z}$, we have $\tilde{x} < \hat{z}$.

Similarly, since $\tilde{x} < 2$, we have $\tilde{x}+2 < 4$ and so $(\tilde{x}+2)/2 < 2$ and thus $\hat{z} < 2$. We have thus shown that $\tilde{x} < \hat{z}$ and $\hat{z} < 2$. Since $\tilde{x}$ was an arbitrary number less than 2, we conclude that for all real numbers x satisfying $x < 2$, there is a real number z such that $x < z$ and $z < 2$. □

Here are some additional remarks about the proof.

- The most important thing to realize about the proof of a universal-existential proposition is that your job is to explain how, given as input an object $\tilde{x}$ that satisfies the hypothesis, it is possible to select an output such that $\tilde{x}$ and your selected output satisfy the conclusion. For this, you must (1) give precise instructions for selecting the output, and (2) carefully prove that $\tilde{x}$ and the number given by your instructions satisfy the conclusion.

- When doing universal-existential proofs, it is usually not at all obvious how to construct the output from the input. It often requires considerable thought and ingenuity. Once you think you know how to choose the output, you should try to prove it. If the proof succeeds, great! If not, then you may have to try another way to choose the output.

- Whenever you are proving the existence of something, there may be more than one choice that satisfies the required conditions. That is true here also. In this case, there are other ways to select the output. Consider the exercise below.

- In this proof, the object $\tilde{x}$ is the hypothetical object which we think of being chosen by someone else. We need this because $\tilde{x}$ corresponds to the value x that appears with a universal quantifier in what we are trying to prove.

  The object $\hat{z}$ (read "z hat") plays a very different role in this proof. It is not a hypothetical object. You should not think of it as being chosen by someone else, but rather as being chosen by the prover of the theorem to help accomplish the goal of the proof. Such an object is referred to as an *auxiliary object*.

In learning how to do proofs, it is very important to keep straight which objects are hypothetical objects and which are auxiliary objects. We will discuss this in much more detail in Supplement 3.

In order to keep from confusing these two types of objects, we use different notation for their names. The name we give to a hypothetical object has a "˜" (tilde) over it. In these notes, the names assigned to auxiliary objects will have a "ˆ" (hat) over them.

Exercise. Redo the above proof with output chosen to be $(2\tilde{x}+6)/5$ instead of $(\tilde{x}+2)/2$. Can you make the proof work with $(3\tilde{x}+6)/5$ instead of $(\tilde{x}+2)/2$?

Example. Here's a similar but more difficult example. Prove: For any positive real number x such that $x^2 < 2$ there is a real number w such that $x < w$ and $w^2 < 2$.

As before, we begin by planning out the proof. Using the hypothetical object method, we'll let $\tilde{x}$ be an arbitrary positive real number satisfying $\tilde{x}^2 < 2$. This tells us that $\tilde{x} < \sqrt{2}$. By an argument similar to the previous proof, we should be able to prove that there is a number z such that $\tilde{x} < z$ and $z < \sqrt{2}$, and we should be able to deduce from this that $z^2 < 2$.

But there is a problem with this approach. The argument is correct, it is incomplete because it uses a fact that we have not yet proved: that 2 has a square root. Since we don't yet know how to prove that 2 has a square root, we must prove our proposition without referring to $\sqrt{2}$.

We are given as input a positive real number $\tilde{x}$ that satisfies $\tilde{x}^2 < 2$. We are looking for a number w that is bigger than $\tilde{x}$ and satisfies $w^2 < 2$. As in the previous proof we want to choose w to be "a little bit bigger" than $\tilde{x}$, but our proof must give precise instructions for choosing such a w. A good way to think about this is to think of the number we will choose as $\tilde{x}$ plus a small amount, and give that small amount a name. It is common to use the Greek letter ε for numbers that we think of as relatively small so let's think of w as $\tilde{x}+\varepsilon$. So we want to choose ε to satisfy the two requirements (i) $\varepsilon > 0$ and (ii) $(\tilde{x}+\varepsilon)^2 < 2$. The second requirement can be rewritten as $(2\tilde{x}+\varepsilon)\varepsilon < 2-\tilde{x}^2$, so we have to choose ε to be small enough to guarantee that the left hand side is less than the positive number $2-\tilde{x}^2$. Now since $\tilde{x}^2 < 2$ we can deduce that $\tilde{x} < 2$ so for any choice of ε, $(2\tilde{x}+\varepsilon)\varepsilon < (4+\varepsilon)\varepsilon$. Provided that we choose $\varepsilon < 1$ this will be less than 5ε. So if we choose $\varepsilon = (2-\tilde{x}^2)/5$, then $(2\tilde{x}+\varepsilon)\varepsilon < 5\varepsilon \leq 2-\tilde{x}^2$ as needed.

Now that we have a good guess for how to define the output from the input, we are ready to try to write our proof.

Proof. Let $\tilde{x}$ be an arbitrary positive real number that satisfies $\tilde{x}^2 < 2$. Our goal is to show that there is a number w such that $w > \tilde{x}$ and $w^2 < 2$. This is the same as showing that there is a positive number ε such that $(\tilde{x}+\varepsilon)^2 < 2$, since if we can find such a ε then $\tilde{x}+\varepsilon > \tilde{x}$ and $(\tilde{x}+\varepsilon)^2 < 2$.

Let $\hat{\varepsilon} = (2-\tilde{x}^2)/5$. Since $2-\tilde{x}^2 > 0$, this is positive. We need to show that $(\tilde{x}+\hat{\varepsilon})^2$ is less than 2.

We have:

$$(\tilde{x}+\hat{\varepsilon})^2 = \tilde{x}^2 + (2\tilde{x}+\hat{\varepsilon})\hat{\varepsilon}.$$

Now, we must have $\tilde{x} < 2$. This is because if, instead, $\tilde{x} \geq 2$ then $(\tilde{x})^2 \geq 4$ and this is impossible, since $\tilde{x}^2 < 2$. Also, $\hat{\varepsilon} < 1$ since $\hat{\varepsilon} = (2 - \tilde{x}^2)/5 \leq 2/5$. Since $\hat{\varepsilon} > 0$ we have that $(2\tilde{x} + \hat{\varepsilon})\hat{\varepsilon} < (2(2) + 1)\hat{\varepsilon} = 5\hat{\varepsilon}$ and so:

$$(\tilde{x} + \hat{\varepsilon})^2 < \tilde{x}^2 + 5\hat{\varepsilon} = \tilde{x}^2 + 5\frac{2 - \tilde{x}^2}{5} = 2.$$

Thus $\tilde{x} + \hat{\varepsilon} > \tilde{x}$ and $(\tilde{x} + \hat{\varepsilon})^2 < 2$, and this demonstrates that there is a number w such that $w > \tilde{x}$ and $w^2 < 2$. Since $\tilde{x}$ was an arbitrary real number satisfying the hypothesis, we conclude that for every positive real number x such that $x^2 < 2$ there is a real number w such that $x < w$ and $w^2 < 2$. □

Example. Often, a proposition is not formulated as a universal-existential proposition, but can be reformulated to be a universal-existential proposition. Let us consider the proposition:

For all odd integers a and b, $a + b$ is even.

As we all know, this is a true universal proposition. Let's see how we would go about proving it. First let's note that the hypothesis is "a and b are each odd integers" and the conclusion is "$a+b$ is even". Now the conclusion $a+b$ is even means "There exists an integer k such that $a+b = 2k$." So the conclusion is an existential predicate. So this is equivalent to a universal-existential proposition.

In order to prove this theorem, we need to have a definition of what it means for an integer to be odd. Here is the definition we use: An integer n is said to be odd provided that there is an integer d such that $n = 2d + 1$.

Proof. Let $\tilde{a}$ and $\tilde{b}$ be arbitrary odd integers. We will show that $\tilde{a} + \tilde{b}$ must be even. By the definition of even, to show $\tilde{a} + \tilde{b}$ is even we must show that there exists an integer k such that $\tilde{a} + \tilde{b} = 2k$.

Since $\tilde{a}$ is odd, by definition of odd, there is an integer, which we will call $\hat{s}$, such that $\tilde{a} = 2\hat{s}+1$. Similarly, $\tilde{b}$ is odd, so there is an integer, which we will call $\hat{t}$, such that $\tilde{b} = 2\hat{t} + 1$. Then

$$\tilde{v} + \tilde{w} = (2\hat{s} + 1) + (2\hat{t} + 1) = 2\hat{s} + 2\hat{t} + 2 = 2(\hat{s} + \hat{t} + 1).$$

Since $\hat{s}$, $\hat{t}$ and 1 are integers, their sum $\hat{s} + \hat{t} + 1$ is also an integer. Therefore we have shown that $\tilde{a} + \hat{b}$ is 2 times the integer $\hat{s} + \hat{t} + 1$ and therefore is even.

Since $\tilde{a}$ and $\tilde{b}$ were chosen to be arbitary odd integers, we conclude that for all odd integers a and b, $a + b$ is even. □

9 Testing a proof using trial values

When solving mathematical problems, we all know the importance of checking your work for errors. Checking proofs is similarly important.

If we have a proof of a universal proposition that uses the hypothetical object method, then one of the most useful ways to check for obvious errors is by using *trial values*. Here's how this works. Suppose we are proving a proposition of the form, "For all x satisfying the hypothesis $H(x)$, x must satisfy the conclusion $C(x)$." We use the hypothetical object method and begin the proof with "Let $\tilde{x}$ be an arbitrary number satisfying $H(\tilde{x})$. We wlll prove $C(\tilde{x})$." To test our proof, we select a *specific value* for $\tilde{x}$ that satisfies H and then substitute that value for $\tilde{x}$ in the main part of the proof. If our proof is correct then it should provide a proof that C holds for the specific value we selected.

This method is especially useful in checking the proof of a universal-existential propostion.

Let's try this method on the previous proof. Here we started by letting $\tilde{a}$ and $\tilde{b}$ be arbitrary odd integers. So let's use the trial values $\tilde{a} = 37$ and $\tilde{b} = 15$. Let's rewrite the main part of the proof with this substitution:

> We will show that $37 + 15$ must be even. By the definition of even, to show $37 + 15$ is even we must show that there exists an integer k such that $37 + 15 = 2k$. We will now give instructions for finding a value for k that satisfies the required conditions.
>
> 37 is odd and the definition of odd tells us that there is an integer, which we will call $\hat{s}$, such that $37 = 2\hat{s} + 1$. Similarly, 15 is odd, so there is an integer, which we will call $\hat{t}$, such that $15 = 2\hat{t} + 1$. Then
>
> $$37 + 15 = (2\hat{s} + 1) + (2\hat{t} + 1) = 2\hat{s} + 2\hat{t} + 2 = 2(\hat{s} + \hat{t} + 1).$$
>
> Since $\hat{s}$, $\hat{t}$ and 1 are integers, their sum $\hat{s} + \hat{t} + 1$ is also an integer. Therefore we have shown that $37 + 15$ is 2 times the integer $\hat{s} + \hat{t} + 1$ and therefore is even.

After this substitution, the proof still makes grammatical sense. Now, let's see if all the steps are correct.

The proof says to choose $\tilde{s}$ to be an integer so that $2\tilde{s} + 1 = 37$. We can select $\tilde{s}$ to be 18, which is an integer as required. Next $\tilde{t}$ is selected so that $2\tilde{t} + 1 = 15$. Here we can select $\tilde{t}$ to be 7, which is an integer as required. Next we check that $2(\tilde{s} + \tilde{t} + 1)$ which is equal to 37+15. Since both quantities are 52, the proof passes our test.

Checking your proof using trial values is a good way to catch errors. Sometimes, when you substitute trial values you see that the result does not make sense, or that there is an obvious error (see the next example). If the proof passes your test, this does not guarantee that the proof is correct, but it provides you with additional confidence of its correctness. You should get in the habit of testing your proofs on at least one (and possibly more) trial values.

Example. Let's return to an earlier example.

> Prove: For all real numbers x, if $x < 2$ then there is a real number z such that $x < z$ and $z < 2$.

Here is a faulty proof.

Proposed proof. Let $\tilde{x}$ be an arbitrary real number satisfying $\tilde{x} < 2$. We will show that there is a real number z such that $\tilde{x} < z$ and $z < 2$. We select $\hat{z}$ to be $\tilde{x} + .00001$.

Since $.00001 > 0$, $\tilde{x} + .00001 > \tilde{x}$. Since $\tilde{x} < 2$ and .00001 is a very small number, $\tilde{x} + .00001$ is still less than 2.

Therefore $\tilde{x} < \tilde{x} + .00001$ and $\tilde{x} + .00001 < 2$. Since $\tilde{x}$ was an arbitrary number less than 2, we conclude that for all real numbers x satisfying $x < 2$, there is a real number z such that $x < z$ and $z < 2$. □

If we use trial values to test this proof, we might try $\tilde{x} = 1.99$. Our instructions say to choose $\hat{z} = 1.99001$ which does satisfy $1.99 < 1.99001$ and $1.99001 < 2$.

But, if we try $\tilde{x} = 1.999999$ then $\tilde{x} + .00001 = 2.000009$ which is bigger than 2. So the proof must be faulty, which means that there must be some error in reasoning. In this case, the error is fairly obvious: just because .00001 is a very small number we can't be sure that $\tilde{x} + .00001$ will be less than 2, because $2 - \tilde{x}$ might be even less than .00001.

10 The hypothetical object method and the textbook

The proofs in the textbook seem to have a different format than those in these notes and students may wonder: "Which way am I supposed to do proofs–the way they are done in the book or the way they are done in the notes?"

The answer to this depends on your instructor. But it is important to realize that the differences between the proofs and those in the notes are minor. While the textbook does not use the terminology "the hypothetical object method", they introduce a very similar method on page 48 "Direct proof of $\forall x, P(x)$".

The main difference between the book's method and the hypothetical object method is that we are more careful about the use of letters in our proofs than the book is. More on this in the next supplement.

11 The standard axioms for the real numbers

We have not exactly specified what facts you can take as given, though we have given some guidelines. The reason for this is that there are too many basic facts that we take as given, and it is difficult to systematically write down all of them. There is another approach. It turns out that for proofs about the real numbers there is is an accepted standard set of axioms, which is given below. This set of axioms is rather small, consisting of only 16 axioms. Nevertheless, these axioms are enough to prove everything you learned in calculus.

We will present this system of axioms below, but we will not spend much time with them. The reason for this is that while we could, if we really wanted to, prove everything from calculus using these axioms, working only from these axioms is very tedious. There are many simple facts, (such as 2+2=4), that are not axioms, and so you would have to prove these facts. The proofs we gave in the previous sections used simple known facts, but many of these facts are not axioms, and if we tried to give complete proofs starting only from the axioms, the proofs would be much longer than they are.

So, while we will take the standard axioms as a starting point, for most of this course we will use the guidelines stated earlier to determine our known facts. But still it is worth spending some time working with the standard axiom system, which we'll do in this section.

The real number system consists of certain mathematical objects, and certain axioms about them. In defining this system, we assume that we already have the concept of "set" as a "collection of objects". Furthermore, when we say that two objects a and b are equal, written $a = b$ we mean that they are the same. This means we have the *substitution principle*: whenever two objects are equal, in any expression involving one of them, we can substitute the other and the resulting expression is equal to the original.

The real number system consists of:

- A set $\mathbb{R}$ whose members are called *real numbers*. The set $\mathbb{R}$ includes two special elements called 0 and 1.
- Two operations $+$ and $\times$ on $\mathbb{R}$. Thus associated to any pair x, y of real numbers there are unique real numbers denoted $x + y$ and $x \times y$.
- There is a relation $<$ on $\mathbb{R}$, which means that for each specific pair of real numbers x, y, "$x < y$" is a mathematical proposition that is either true or false.

The axioms of the real number system are classified into the groups: *Algebraic Axioms*, *Order Axioms* and the *Completeness Axiom*.

Algebraic axioms

Commutativity of $+$	For all $x, y \in \mathbb{R}$, $x + y = y + x$.
Associativity of $+$	For all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
Commutativity of $\times$	For all $x, y \in \mathbb{R}$, $x \times y = y \times x$.
Associativity of $\times$	For all $x, y, z \in \mathbb{R}$, $x \times (y \times z) = (x \times y) \times z$.
Distributive Law	For all $x, y, z \in \mathbb{R}$, $x \times (y + z) = x \times y + x \times z$.
Axiom of 0	For all $x \in \mathbb{R}$, $x + 0 = x$.
Axiom of 1	For all $x \in \mathbb{R}$, $x \times 1 = x$.
Nontriviality	$0 \neq 1$.
Existence of additive inverses	For all $x \in \mathbb{R}$, there is a $y \in \mathbb{R}$ such that both $x + y = 0$ and $y + x = 0$.
Existence of multiplicative inverses	For all $x \in \mathbb{R}$, if $x \neq 0$ then there is a $y \in \mathbb{R}$ such that both $x \times y = 1$ and $y \times x = 1$.

Order axioms

Irreflexivity of $<$	For all $x \in \mathbb{R}$, $\sim (x < x)$.
Transitivity of $<$	For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$ then $x < z$.
Trichotomy axiom	For all $x, y \in \mathbb{R}$ exactly one of the relations $x < y$, $x = y$ and $y < x$ holds.
Addition property of $<$.	For all $x, y, z \in \mathbb{R}$, if $x < y$ then $x + z < y + z$.
Multiplication property of $<$.	For all $x, y, z \in \mathbb{R}$ if $x < y$ and $z > 0$ then $x \times z < y \times z$.

All of the above axioms state properties with which you are probably very familiar. However, there is one additional axiom that is very different from the rest, and takes some thought to understand.

There are various ways to formulate the Completeness Axiom, and the version given below is different from those you will see in most places (including the textbook), but is probably the easiest version to understand. While the other axioms talk about properties of one, two or three arbitrary real numbers, this axiom talks about two *subsets* of $\mathbb{R}$.

It is not necessary that you completely understand this axiom at this point in the course; we will come back to it later.

Completeness Axiom (Non-standard version) For any two nonempty subsets A and B of the set $\mathbb{R}$, if for all $x \in A$ and $y \in B$, $x < y$ then there is a real number z such that for every $x \in A$, $x < z$ or $x = z$ and for every $y \in B$, $z < y$ or $z = y$.

Okay, so these are the standard concepts and axioms for the real numbers. One thing to notice is that there are many things missing. For example, there is no mention of the numbers 2 or 10 or π, or subtraction or division, or the relation $\leq$. There is no mention of positive and negative numbers. There are lots of simple facts that are not axioms such as:

- $2 + 2 = 4$.
- For all real numbers x, $x \times 0 = 0$.
- For all real numbers x, y, if $xy = 0$ then $x = 0$ or $y = 0$.
- For all real numbers x, y, $(-x) \times y = x \times (-y)$.
- For all real numbers x, y, $(x + y)^2 = x^2 + 2xy + y^2$.
- For all real numbers x, there is exactly one real number y such that $x + y = 0$.

So, what we call the real number system is missing lots of important things. Nevertheless, using these axioms, together with the basic notion of what a set is and what a function is, one can prove everything you learned in calculus!

To do this, one needs to build up new concepts (definitions) from the basic concepts of $+, \times, 0, 1$ and $<$.

For example we make the following definitions:

Definitions

1. (Definition of *the number* 2) Since 1 is a real number, we can add 1 to itself to get the number $1 + 1$. We *define* the number 2 to be $1 + 1$.
2. (Definition of *the numbers* 3 *and* 4) Similarly we define the number 3 to be $2 + 1$, and the number 4 to be $3 + 1$.
3. (Definition of *positive number* and *negative number*) An arbitrary real number x is said to be a *positive number* provided that $x > 0$ and is said to be a *negative number* provided that $x < 0$.
4. We define $\leq$ to be the relation on $\mathbb{R}$ as follows: for $x, y \in \mathbb{R}$, $x \leq y$ means $x < y$ or $x = y$.

Now let's prove some Theorems:

Theorem 1. 2+2=4.

Of course, it seems silly to be proving this, but it gives us a chance to see how the definitions and axioms can be used.

Proof. By the definition of 4, 4=3+1. By the definition of 3, 3=2+1. Therefore 4=3+1=(2+1)+1. Now, by the associativity of addition, (2+1)+1=2+(1+1), and by definition of 2, this equals 2+2. Therefore, 4=2+2. □

Here are more examples.

Theorem 2. For all $x, y, z \in \mathbb{R}$, if $x \leq y$ then $x + z \leq y + z$.

Let's first note that, while similar to the addition property of $<$, it is not the same since it uses $\leq$ rather than $<$. Here's a proof, using the hypothetical object method.

Proof Let $\tilde{x}, \tilde{y}$ and $\tilde{z}$ be arbitrary real numbers satisfying $\tilde{x} \leq \tilde{y}$. We must show $\tilde{x} + \tilde{z} \leq \tilde{y} + \tilde{z}$. By definition of $\leq$, $\tilde{x} \leq \tilde{y}$ means that $\tilde{x} < \tilde{y}$ or $\tilde{x} = \tilde{y}$. We consider these two possibilities separately.

Case 1. Assume $\tilde{x} < \tilde{y}$. Then by the addition property of $<$, $\tilde{x} + \tilde{z} < \tilde{x} + \tilde{z}$, and therefore $\tilde{x} + \tilde{z} \leq \tilde{x} + \tilde{z}$.

Case 2. Assume $\tilde{x} = \tilde{y}$. Then $\tilde{x} + \tilde{z} = \tilde{y} + \tilde{z}$ by the substitution principle and so $\tilde{x} + \tilde{z} \leq \tilde{y} + \tilde{z}$.

In either case, we have the desired conclusion, so since $\tilde{x}, \tilde{y}, \tilde{z}$ were arbitrary numbers satisfying the hypothesis we conclude that for all $x, y, z \in \mathbb{R}$, if $x \leq y$ then $x + z \leq y + z$. □

Theorem 3. For all real numbers x, there is a unique real number y such that $x + y = 0$.

Proof. Let $\tilde{x}$ be arbitrary. There are two things to show:

1. There is at least one real number y such that $\tilde{x} + y = 0$.
2. For any two real numbers y, z if $\tilde{x} + y = 0$ and $\tilde{x} + z = 0$ then $y = z$.

The first part follows immediately from the existence of additive inverses.

To prove the second part, let $\tilde{y}$ and $\tilde{z}$ be arbitrary real numbers satisfying $\tilde{x} + \tilde{y} = 0$ and $\tilde{x} + \tilde{z} = 0$. By commutativity of addition, we also have $\tilde{z} + \tilde{x} = 0$. We must show that $\tilde{y} = \tilde{z}$. First we have:

$$\begin{aligned}(\tilde{z} + \tilde{x}) + \tilde{y} &= 0 + \tilde{y} \\ &= \tilde{y} \qquad \text{by the axiom of 0.}\end{aligned}$$

We also have:

$$\begin{aligned}(\tilde{z} + \tilde{x}) + \tilde{y} &= \tilde{z} + (\tilde{x} + \tilde{y}) \quad \text{by the associativity of addition} \\ &= \tilde{z} + 0 \\ &= \tilde{z} \qquad \text{by the axiom of 0.}\end{aligned}$$

Combining these two equations gives that $\tilde{y} = \tilde{z}$. □

Based on this theorem, we can make the following definition: For any real number x, we define the notation $-x$ to be mean the unique real number such that $x + (-x) = 0$. This number is called the *additive inverse* of x.