9 Binary relations and binary relationships [10]

Binary relationships When studying mathematical objects, the mathematical scenario being considered may specify conditions (requirements) on two or more objects at a time. A condition on two mathematical objects is called a *binary relationship*.

Example 9.1. Here are some relationships that might hold between two real numbers x and y:

x **is less than** y**,** abbreviated $x < y$. Similarly $x \leq y$, $x \geq y$ and $x > y$.

y **is the square of** x**,** which is abbreviated by $y = x^2$,

x **is within** ε **of** y**.** Here $\varepsilon \geq 0$ and this relationship means $|x - y| \leq \varepsilon$.

y **is the image of** x **under** f**,** where f is a function from $\mathbb{R}$ to $\mathbb{R}$. This relationship is abbreviated $y = f(x)$. Note that the relationship "y is the square of x" is a special case of this one.

Any equation or inequality involving x and y, such as "$|x - y| \leq 1$", "$x^2 + y^3 = 1$" or "$(x + y)^2 \leq x^3$" defines a relationship between x and y.

Example 9.2. Here are some relationships that might hold between two integers m and n:

m **is the successor of** n**,** which means $m = n + 1$.

m **is a divisor on** n**,** abbreviated $m|n$, is the requirement that n/m is an integer.

m **is coprime to** n**,** which means that m and n have no common divisor greater than 1.

m **and** n **differ by a multiple of** k (where k is some integer), which is abbreviated $m \equiv_k n$.

Example 9.3. Here are some relationships that might hold between two sets A and B

A **is a subset of** B**, abbreviated** $A \subseteq B$

A **is disjoint from** B**,** which means $A \cap B = \emptyset$.

A **and** B **have the same size,** abbreviated $|A| = |B|$.

Example 9.4. Here are some relationships that might hold between two lists k and ℓ:

k **is a rearrangement of** ℓ**,** provided that every object that appears in either k or ℓ, appears as an entry of both lists the same number of times. For example $(1, 1, 1, 2, 2, 3)$ is a rearrangement of $1, 2, 3, 1, 2, 1)$.

k **is a prefix of** ℓ**.** This means that there is a list k' such that $\ell = k * k'$, where $*$ is the concatentation operation.

[10]Version 3-29-2015.

Example 9.5. In the previous examples, the relationships considered are between objects of the same type. We can also consider relationships between objects of different types:

For a list L of integers and an integer a, we may have the relationship "a is an entry of L".

For a function f from A to B and object $b \in B$, we may have the relationship "b is in the range of f".

For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $b \in \mathbb{R}$ we say that b *is a root of* f if $f(b) = 0$.

Technically, a binary relationship is a mathematical scenario involving two objects one specified types S and T (where possibly $S = T$). To specify the relationship we specify the types of the two objects, and the constraint (indefinite assertion) that they must satisfy.

As with any mathematical scenario, we can consider the *set of feasible instances* of the scenario. The feasible instances are *ordered pairs* of values (s, t) from $S \times T$ that satisfy the constraint specified by the relationship. For example, the feasible instances of the "less than" relationship for real numbers is the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$.

A subset of $S \times T$ is called a *binary relation over* $S \times T$. Thus each binary relationship (mathematical scenario) over $S \times T$ corresponds to a binary relation over $S \times T$ which is a subset of $S \times T$.

Binary relations

Definition 9.1. A binary relation R is a mathematical object consisting of:

- A set called the*source* of R, denoted $\mathbf{source}(R)$
- A set called the *target* of R, denoted $\mathbf{target}(R)$
- A set of ordered pairs that is a subset of $\mathbf{source}(R) \times \mathbf{target}(R)$.

The set of ordered pairs is the most important component of R. When we introduce a relation R as a relation *on* $S \times T$ or as a relation *from* S *to* T. we mean that $\mathbf{source}(R) = S$ and $\mathbf{target}(R) = T$.

Two relations R and Q are equal if they have the same source, target and pairs. It's almost always clear from the definitions of R and Q whether they have the same source and target, and if they have the same source and target we'll usually skip this fact in proving that $Q = R$.

Relation symbol For $x \in \mathbf{source}R$ and $y \in \mathbf{target}R$, the sentence "(x, y) is an ordered pair in the relation" can be abbreviated by $(x, y) \in R$ or by the notation xRy. We also write $x\not{R}y$ if $(x, y) \notin R$. Sometimes we introduce a special symbol such as $\sim$, $\equiv$ or $<$ to denote the relationship.

Reverse relations If R is a relation from S to T, the *reverse relation* $\overleftarrow{R}$ is the relation from T to S with pair set $\{(t, s) : (s, t) \in R\}$

Image of a set, and reverse image If R is a relation from S to T and $X \subseteq S$, we define the image of X under R, written $R(X)$, to be the set $\{t \in T : \exists x \in X \text{ such that } xRt\}$.

Similarly if $Y \subseteq T$, the *reverse image of T under R or the image of T under $\overleftarrow{R}$, written $\overleftarrow{R}(T)$, is the set $\{s \in S : \exists y \in Y \text{ such that } sRy\}$.*

Operations on Relations *Since relations are (basically) sets of ordered pairs, we can take the union, intersection and difference of two relations. We have to be a little careful in defining the source and target of the combined relation. We use the following convention: when we combine relations (by union, intersection and difference), the source of the combined relation is the union of the sources of all of the relations being combined and the target of the combined relation is the union of the targets of all relations being combined.*

9.1 Relations on a set

Many interesting relations have the same source and target.

Definition 9.2. A *relation on set A* is a relation whose source and target is A.

One of the simplest relations on a set A is the diagonal relation *on A, denoted D_A (or simply D if A is understood from context.)* $D_A = \{(x, x) : x \in A\}$.

We classify relations on a set according to certain basic properties.

Definition 9.3. For a set A and relation R on A we say that R is

reflexive provided that for all $a \in A$ aRa.

antireflexive provided that for all $a \in A$, $\neg(aRa)$.

symmetric provided that for any $a, b \in A$, aRb implies bRa.

antisymmetric provided that for any $a, b \in A$ if $a \neq b$ and aRb then $\neg(bRa)$.

transitive provided that for any $a, b, c, \in A$, if aRb and bRc then aRc.

full provided that for any $a, b \in R$ with $a \neq b$, aRb or bBa.

Exercise 9.1. Suppose R is a transitive relation.

1. Prove that for any x and y if xRy then $R(y) \subseteq R(x)$.

2. Suppose that x and y are elements such that $R(y) \subseteq R(x)$. Does this necessarily imply that xRy?

Two important types of relations on a set *There are two types of relations that arise very frequently in mathematics. these are* equivalence relations *and* partial order relations *which we now describe.*

Equivalence relations *In many situations we classify objects of a particular type into groups of "similar" objects. We might classify real numbers into three types "positive", "negative" and "zero", In geometry we classify triangles according to their sequence of angles (such as 30,60,90 triangles) and two triangles with the same sequence of angles are said to similar. We can classify finite sets according to the number of elements they have; we say that two sets are* equal-sized *if they have the same number of elements.*

In such a relation, the set of objects A being classified is partitioned into classes *and the relation consists of all pairs of objects that belong to the same class.*

Definition 9.4. Equivalence relations. Suppose X is a set and let $\mathcal{P}$ be a partition of X. The *equivalence relation $R_{\mathcal{P}}$ induced by the partition $\mathcal{P}$* is the set of pairs $(x, y) \in X \times X$ such that x and y are in the same part of $\mathcal{P}$. The parts of the partition are called the *equivalence classes* of the equivalence relation.

Equivalence relations have a very simple structure: once you know the partition it's easy to see which elements are related to each other.

Example 9.6.

- Take the set of all lists. Every list has an associated set of the entries that appear. For example, the set associated to the list $(1, 3, 5, 3, 6, 3, 1)$ is $\{1, 3, 5, 6\}$. If we classify lists according to the associated set of entries we get an equivalence relation on lists.
- We can also classify lists according to their lengths so that all lists with the same length are in the same equivalence class.
- Consider what happens when you round a real number to the closest integer. This rounding rule is ambiguous for numbers like 3.5 since it is equally close to 3 and 4; in this case let's say we round the integer down. Let's say that two real numbers are *approximately equal* if they round to the same integer, so 8.8 and 9.3 are approximately equal. The relationship *approximately equal* defines an equivalence relation.
- Let's classify integers according to the largest power of 2 they are divisible by. Thus 24 and 40 are divisible by 8 but not 16 and are classified the same.

Exercise 9.2. Precisely describe the partition into equivalence classes for the approximately equal relation.

Each of these equivalence relations is obtained by taking a set and partitioning it into classes, called equivalence classes.

Which of the properties does an equivalence relation satisfy.

Theorem 9.1. *Every equivalence relation is reflexive, symmetric and transitive.*

Proof. Suppose X is an arbitrary set and $\mathcal{P}$ is an arbitrary partition and let R be the associated equivalence relation. We claim that R is reflexive, symmetric and transitive.

Proof that R is reflexive. Suppose $x \in X$ is arbitrary. Then x belongs to exactly one set $\mathcal{P}$ and so xRx.

Proof that R is symmetric. Suppose x, y are arbitrary members of X. Assume xRy. Then there is a set of $\mathcal{P}$ that contains both x and y so also yRx.

Proof that R is transitive. Suppose x, y, z are arbitrary members of X. Assume xRy and yRz. Then there are sets $A, B \in \mathcal{P}$ such that $x, y \in A$ and $y, z \in B$. Since $y \in A \cap B$ and $\mathcal{P}$ is a partition, we have $A = B$. Then $x, z\ inA$ and so xRz. □

Remarkably it turns out that the converse of Theorem 9.1 is true. Namely if we have any relation that is known to be reflexive, symmetric and transitive, then it must be an equivalence relation.

Theorem 9.2. *Suppose R is a relation on X that is transitive, symmetric, and reflexive. Then there is a partition $\mathcal{P}$ of X such that $R = R_{\mathcal{P}}$.*

Remark 9.1. We defined an equivalence relation to be a relation that comes from a partition by saying two objects are related if they belong to the same part of the partition. Another way to define equivalence relation (which appears in many other books) is to say that an equivalence relation is a relation that is transitive, symmetric and reflexive. We refer to the first definition as the *partition-based* definition of equivalence relations and the second definition as the *property-based* definition. Theorems 9.1 and 9.2 together imply that these two very different definitions lead to the same thing.

Proof. Suppose R is an equivalence relation on X.

Let $\mathcal{P} = \{R(\{x\}) : x \in X\}$. We claim that $\mathcal{P}$ is a partition of X and that $R = R_{\mathcal{P}}$.

First we prove that $\mathcal{P}$ is a partition of X. We must show that (i) for all $x \in X$, there is a member of $\mathcal{P}$ that has x as a member, and (ii) for all $A, B \in \mathcal{P}$ if $A \cap B \neq \emptyset$ then $A = B$.

Proof of (i). Suppose $x \in X$ is arbitrary. Then $R(\{x\}) \in \mathcal{P}$. Since R is reflexive $x \in R(\{x\})$.

Proof of (ii). Suppose $A, B \in \mathcal{P}$. Assume that $A \cap B \neq \emptyset$. We must show $A = B$. For this we must show $A \subseteq B$ and $B \subseteq A$. By definition of $\mathcal{P}$ there are objects we'll call $a, b \in X$ such that $A = R(\{a\})$ and $B = R(\{b\})$.

Proof that $A \subseteq B$. Let $x \in A$ be arbitrary. Then by definition of $A = R(\{a\})$, we have aRx. We must show $x \in B$, which means we must show bRx. Since $A \cap B \neq \emptyset$ there is an object we'll call y such that $y \in A$ and $y \in B$ and thus aRy and bRy. By symmetry of R we have yRa and by transitivity bRy and yRa implies bRa. Since bRa and aRx we have bRx by transitivity, as required to show $A \subseteq B$.

The proof that $B \subseteq A$ is analogous by interchanging A and B and a and b in the above proof.

So we've shown that $\mathcal{P}$ is a partition. It remains to show that $R = R_{\mathcal{P}}$. For this we must show that for all $x, y \in X$ xRy if and only if $xR_{\mathcal{P}}y$. Suppose $x, y \in A$. We must prove two things: (a) if xRy then $xR_{\mathcal{P}}y$ and (b) if $xR_{\mathcal{P}}y$ then xRy.

Part (a). Assume xRy. Then $y \in R(\{x\})$ and also $x \in R(\{x\})$ and so $x, y \in R(\{x\})$ and so $xR_{\mathcal{P}}y$.

Part (b). Assume $xR_{\mathcal{P}}y$. Then there is a set $S \in \mathcal{P}$ that contains both x and y. Now $x \in R(\{x\}) \in \mathcal{P}$ and so $R(\{x\}) \cap S \neq \emptyset$. By the definition of a partition, $S = R(\{x\})$ and since $y \in S$ we have $y \in R(\{x\})$ and so xRy as required. □

Given an equivalence relation R the parts of the associated partition $\mathcal{P}_R$ are called the equivalence classes *of R. The proof of the above theorem shows:*

Porism 9.3. *For any equivalence relation R on X, the set $\{R(\{a\}) : a \in X\}$ is the partition of X into equivalence classes of R. For any two members a, b of X we either have $R(\{a\}) = R(\{b\})$ or $R(\{a\}) \cap R(\{b\}) = \emptyset$.*

Remark 9.2. A *porism* is another name for a proven assertion, that is similar to *corollary.* Recall that a *corollary* of a theorem is an assertion that is proved fairly easily from the theorem. A *porism* of a theorem is not proved directly from the theorem, but rather is a consequence of the *given proof* of the theorem. In this case, our proof of Theorem 9.2 explicitly proves the Porism.

The importance of Theorem 9.2 is that it happens often in mathematics, that we have a relation on a set X where we can prove that it is en equivalence relation by showing that it is reflexive, symmetric and transitive, even though we don't know the equivalence classes. By studying the relation and using Porism 9.3 we can figure out the equivalence classes. We know look at an important example where this happens.

Equivalence modulo an integer k *Suppose k is a positive integer. For integers x and y we say that x* is equivalent to y modulo k, *or x* is equivalent to y mod k, *written $x \equiv_{(k)} y$, if $x - y$ is divisible by k. We say*

Exercise 9.3. Prove that equivalence modulo k is a reflexive, symmetric and transitive relation on $\mathbb{Z}$.

By Theorem 9.2, there is an associated partition into equivalence classes. We refer to these classes as the mod k equivalence classes). We now describe these equivalence classes.

Theorem 9.4. *For j between 0 and $k-1$ let $C_j = \{j + ak : a \in \mathbb{Z}\}$. Then the C_j are the equivalence classes with respect to $\equiv_{(k)}$.*

Proof. Let R denote the relation consisting of pairs $\{(i,j) : i \equiv_k j\}$. We need to show (1) each set for all $j \in \{0, 1, \ldots, k-1\}$, C_j is an equivalence class of R, and (2) every equivalence class of R is equal to C_j for some $j \in \{0, 1 \ldots, k-1\}$.

Proof of (1): Suppose $j \in \{0, 1, \ldots, k-1\}$ is arbitrary. We claim that $R(\{j\}) = C_j$.

Proof of $R(\{j\}) \subseteq C_j$: Let $s \in R(\{j\})$ be arbitrary. Then $s - j$ is divisible by k so $b = (s-j)/k$ is an integer Thus $s = bk + j$ and so $s \in C_j$.

Proof of $C_j \subseteq R(\{j\})$. Let $s \in C_j$ be arbitrary. Then by definition of C_j there is an integer a such that $j + ak$, so $(s - j$ and so $s - j$ is divisible by k and so $s \in R(\{j\})$.

This completes the proof that each of the sets C_j for $j \in \{0, 1, \ldots, k-1\}$ is an equivalence class.

Proof of (2): Now we want to prove that the sets C_j for $j \in \{0, 1, \ldots, k-1\}$ are the only equivalence classes. Since the equivalence classes partition $\mathbb{Z}$ it suffices to show that every integer lies in one of the sets $\{C_j : j \in \{0, 1 \ldots, k-1\}$.

Let b be an arbitrary integer. We want to show that there is a $j \in \{0, 1 \ldots, k-1\}$ such that $(b-j)$ is divisible by k. Our goal is to define j to be the smallest positive integer such that $b-j$ is divisible by k. This is possible provided that there is at least one positive integer j such that $b-j$ is divisible by k. Let W be the set of all nonnegative integers m such that $b-m$ is divisible by k. We claim that W is nonempty. In the case that $b \geq 0$ we know b itself is in W. In the case $b < 0$ we know that $(1-k)b \in W$ since $(1-k) \leq 0$ and so $(1-k)b \geq 0$ and $b-(1-k)b = kb$ is divisible by k.

Since W is a subset of the nonnegative integers that is nonempty it has a smallest member. Let j be the smallest member of W. Clearly $j \geq 0$. We also claim that $j < k$. Suppose for contradiction that $j \geq k$. Let $j' = j-k$. Then $j' \in W$ since $j' \geq 0$ and $b-j' = (b-j)+k$ is divisible by k since $m-j$ is. But this contradicts the choice of j as the smallest member of W. So $j < k$.

So we have $b \in C_j$ where $j \in \{0, \ldots, k-1\}$ as required. □

Corollary 9.5. *Suppose k is an arbitrary positive integer. For every positive integer b there is a unique integer w that is a multiple of k such that $b-k < w \leq b$.*

Proof. Suppose k is a positive integer and suppose that b is an integer. Let $\{C_0, \ldots, C_{k-1}\}$ denote the equivalence classes modulo k as given by Theorem 9.4. There is an integer that we will call j such that $w \in \{0, 1, \ldots, k-1\}$ and $b \in C_j$. Let $w = b-j$. Since $b \equiv_k j$ we have that w is a multiple of k and $w = b-j > b-k$ and $w \leq b$ as required. □

Example 9.7. Some additional examples of relations that are reflexive, symmetric and transitive (and are therefore equivalence relations).

- Real Numbers under rational equivalence xRy if there are integers a, b such that $ax = by$.
- Integers under odd multiple equivalence: xRy if there are odd integers a, b such that $ax = by$.
- Integers under *square equivalence* xRy: if there are integers a, b such that $xa^2 = yb^2$.
- Given a set S of real numbers, relation on S: xRy if the interval $I[x, y] \subseteq S$ (where $I[x, y] = [\min(x, y), \max(x, y)]$.)
- Ordered pairs of real numbers with $(a, b)R(c, d)$ if and only if $ad = bc$.

Exercise 9.4. For each relation in Example 9.7, prove that the relation is reflexive, symmetric and transitive.

Exercise 9.5. For parts 2,3 and 5 of Example 9.7 determine a simple description of the equivalence classes and prove that your description is correct.

Partial order relations The second important class of relations on a set is the class of *partial orders*. A partial order arises when we have a set, and a way of comparing members where we say that one element x is, in some sense, "greater than" another. The obvious example is the set of real numbers where "greater than" has its usual meaning. Another important example is if we take $\mathcal{S}$ to be all subsets of a set Y and consider the "$\subseteq$" relationship on $\mathcal{S}$. We think of a set A as "greater than" a set B if $B \subseteq A$. Notice that in the real number example, for any two different real numbers x and y exactly one of the statements "$x \leq y$" and "$y \leq x$" is true, while in the sets example, we may have two sets A and B where it is neither the case that $A \subseteq B$ nor $B \subseteq A$.

With these examples in mind, we define a partial order on a set X to be a relation R on X that is *anti-symmetric* and *transitive*, and *reflexive*

Example 9.8. Here are a few basic partial orders:

Real numbers under $\leq$

Subsets under $\subseteq$

Positive Integers under divisibility The relationship $m|n$ divides a partial order on the positive integers.

Lists under the prefix ordering which was defined earlier.

If R is a partial order on X, then two members x and y of X are said to be *comparable* if $x \leq y$ or $y \leq x$ and said to be *incomparable* if neither $x \leq y$ nor $y \leq x$.

Definition 9.5. A partial order on X with the property that any two members of X are comparable is called a *total order* or *linear order* on X.

For example, the $\leq$ relation on $\mathbb{R}$ is a total order but the divisibility order on integers is not.

Our definition of a partial order R on X requires that the relation be reflexive so that xRx for all $x \in X$. We could also consider relations that are transitive, anti-symmetric and *anti-reflexive*. Such a relation is called a *strict partial order*.

For example, strict inequality "$<$" on $\mathbb{R}$ and strict containment $\subset$ on sets are both strict partial orders.

Strict partial orders are very similar to partial orders. In fact, given any strict partial order S on A we get a partial order on A by taking the union of S with the diagonal relation D_A. Similarly we can transform a partial order into a strict partial order by removing all of the pairs from D_A.

Notation for partial orders When we are talking about a fixed partial order R on a set A we often use the notation $x \leq_R y$ to mean xRy, and $x <_R y$ to mean $(xRy) \wedge (x \neq y)$. The symbols $\leq_R$ and $<_R$ emphasize that R is a partial order.

Furthermore, if we are talking about a single partial order R on set A we may omit the subscript "R" and write simply $x \leq y$ and $x < y$. We can do this provided that we are careful not to do this when there is a risk of confusion with the usual meaning of $\leq$.

When introducing a partial order we may say:

- Let R be a partial order on A, or
- Let R be a partial order on A with relation symbol $\leq$

Some special subsets of a partially ordered sets.

Definition 9.6. Suppose R is a partial order on A and $x, y \in A$ with symbol $\leq_R$. We define:

- $[x, y]$ to be the set of elements z such that $x \leq_R z$ and $z \leq_R y$. This is the *closed interval from x to y.*
- $(x, y]$ to be the set of elements z such that $x <_R y$ and $y \leq_R z$. This is the *upper-closed interval from x to y.*
- $[x, y)$ to be the set of elements z such that $x \leq_R y$ and $y <_R z$. This is the *lower-closed interval from x to y.*
- (x, y) to be the set of elements z such that $x <_R y$ and $y <_R z$. This is the *open interval from x to y.*
- $[x, \uparrow)$ to be the set of z such that $x \leq_R z$.
- $(x, \uparrow)$ to be the set of z such that $x <_R z$.
- $(\downarrow, x]$ to be the set of z such that $z \leq_R x$.
- $(\downarrow, x)$ to be the set of z such that $z <_R x$

A subset of R of this form is called an *interval of R*, or simply, an *interval.*

Proposition 9.6. *For the partial order R on A and $x, y \in A$:*

1. $[x, x] = \{x\}$ *and* $[x, x) = (x, x] = (x, x) = \emptyset$.
2. *If x is not less than or equal to y then* $[x, y] = [x, y) = (x, y] = (x, y) = \emptyset$.

Exercise 9.6. Prove Proposition 9.6.

Example 9.9. Intervals of $(\mathbb{R}, \leq)$. For real numbers x, y, $[x, y]$, $[x, y)$, etc. have their usual meanings $[x, \uparrow)$ is the real interval $[x, \infty)$ and $(\downarrow, x)$ is the interval (∞, x).

Intervals of $(\mathbb{Z}, \leq)$ The integer interval $[x, y]$ is obtained by taking the real inteval $[x, y]$ and taking its intersection with $\mathbb{Z}$. Note that for integer intervals $(x, y) = [x + 1, y) = (x, y - 1] = [x + 1, y - 1]$.

Intervals of $(\mathbb{R} \times \mathbb{R}, \leq)$. Elements of this partial order are of the form $(a, b) \in \mathbb{R}^2$

Intervals of $(\mathbb{N}, |)$.

Intervals of $(\mathcal{P}(S) :\subseteq)$.

Definition 9.7. Suppose R is a partial order on the set A. Suppose X is a subset of A. We say that:

- X s an *up-set* (with respect to R) if for all $x, y \in A$ if $x \in X$ and xRy then $y \in X$.
- X is a *down-set* (with respect to R) if for all $x, y \in A$ if $x \in X$ and yRx then $y \in X$.
- X is *order-convex* (with respect to R) if and only if for all $x, y, z \in A$ if $x, z \in X$ and xRy and yRz then $y \in X$.

Upper and lower bounds Suppose R is a partial order on A. If xRy we say that x is *less than or equal to y under R* and y is *greater than or equal to x under R*. If the relation R is clear from the context we may drop the phrase "under R".

Definition 9.8. Suppose R is a partial order on A and $X \subseteq A$.

- A *lower bound for X* is an element of A that is less than or equal to every member of X. We say that X is *bounded below* by z if z is a lower bound for X.
- An *upper bound for X* is an element that is greater than or equal to every member of X. We say that X is *bounded above* by element y if y is an upper bound for X.
- A member $x \in X$ is said to be *minimal in X* if there is no member of X that is strictly less than x.
- A member $x \in X$ is said to be a *minimum of X* if for all $y \in X$, $x \leq y$.
- A member $x \in X$ is said to be *maximal in X* if for all $y \in X$, there is no member of X that is strictly greater than x.
- A member $x \in X$ is said to be a *maximum of X* if for all $y \in X$, $x \geq y$.

Example 9.10. 1. Consider the set $\mathbb{R}$ with the usual order. For the subset $\{-100, 0, 100, 200, 300\}$, any number less than or equal to -100 is a lower bound and any number greater than or equal to 300 is an upper bound. Also -100 is a minimum and also is minimal and 100 is a maximum and is maximal. For the open interval $(0, 1]$ we have that any number less than or equal to 0 is a lower bound and any number greater than or equal to 1 is an upper bound. The set has no minimum or minimal element; the number 1 is both a maximal element and a maximum of the set.

2. Consider the set $\cap P(\mathbb{Z})$ of all subsets of the set of integers with the containment order. For the subset $\{\{1,3\},\{1,3,5\},\{1,3,6,7\}\}$, any subset of $\{1,3\}$ is a lower bound and any superset of $\{1,3,5,6,7\}$ is an upper bound. The set $\{1,3\}$ is a minimum for the subset and also minimal. The sets $\{1,3,5\}$ and $\{1,3,6,7\}$ are maximal elements of the subset but the subset has no maximum.

Proposition 9.7. *Suppose R is a partial order relation on X and suppose S is a subset of X and $x \in X$.*

1. *The element x is a minimum for S if and only if $x \in S$ and x is a lower bound for S.*
2. *The element x is a maximum for S if and only if $x \in S$ and x is an upper bound for S.*

Exercise 9.7. Prove Proposition 9.7.

Exercise 9.8. 1. Prove that for any partial order R on set X and subset S of X,

   (a) Any minimum element of S is also minimal, and
   (b) If S has a minimum element then it is the unique minimal element.

2. Prove or give a counterexample: For any partial order R on set X and subsets S of X, if x is the unique minimal element of S then x is a minimum of S.

Definition 9.9. Suppose R is a partial order on the set A. For $X \subset A$, let $UB(X)$ be the set of upper bounds of X and let $LB(X)$ be the set of lower bounds of X.

Theorem 9.8. *Let S be a set and consider the powerset $\mathcal{P}(S)$ as a partially ordered set under the inclusion order. Then for any $\mathcal{B} \subseteq \mathcal{P}(S)$:*

$$\begin{aligned} UB(\mathcal{B} &= [\bigcup \mathcal{B}, \uparrow). \\ LB(\mathcal{B} &= (\downarrow, \bigcap \mathcal{B}]. \end{aligned}$$

Exercise 9.9. Prove Theorem 9.8.

Exercise 9.10. Let R be a relation on X. Let us say that two elements x and y are R-twins if for every $z \in X$ we have xRz if and only if yRz and zRx if and only if zRy. Let $T = T(R)$ be the relation on X where xTy if x and y are R-twins.

1. Prove that T is an equivalence relation on X.
2. Give an example of a reflexive relation R on a 6 element set for which $T(R)$ has two equivalence classes of size 3.
3. Suppose R is the divisibility relation on $\mathbb{Z}$. What is $T(R)$?