8 Using and abusing variables [9]

One fundamental difference between mathematical communication and most non-mathematical communication, is the pervasive use of variables: letters, subscripted letter, or other symbols that represent mathematical objects that are unspecified or not fully specified. Variables provide the skilled mathematical communicator with an enormously powerful tool to communicate complex ideas effectively. But when used improperly, variables can render mathematical communication incomprehensible or nonsensical.

The use of variables is governed by rules that are intended to ensure that the resulting communication is correct, meaningful and understandable. Failure to follow these rules can cause great confusion, and result in serious errors such as claiming that you've proved something that's false.

As with other proof-writing guidelines presented in this course, the guidelines for using variables are stricter than those used by mathematicians and in most other textbooks. Following the guidelines will protect you against incorrect arguments, and should be followed until you have enough experience to reliably determine when the rules can be safely relaxed.

Which letters are used when? In principle, you can use any letter to represent any mathematical object, but there are some conventions that are commonly used:

- Lower case letters are used for numbers.
- When naming real numbers, it is common to use letters towards the end of the alphabet, especially x, y, z.
- When naming integers, it is common to avoid letters near the end of the alphabet. Commonly used letters for integers include $a, b, c, d, i, j, k, m, n, p, q, r$.
- Sets are usually assigned upper case letter names.
- Functions are often assigned one of the lower case letters f, g or h.

In some cases, there are objects that are so useful that mathematicians have agreed that certain letters will stand for them. Thus $\mathbb{R}$ is used to denote the set of real numbers. Also the letter e is the common abbreviation for the number that is equal to $\lim_{n\to\infty}(1 + 1/n)^n$, and π is the common abbreviation for the number that is the ratio of the circumference of a circle to its diameter. You should only use these letters to mean other things if there is no chance for confusion with their standard mathematical meanings.

The life span of an object: primary objects and dummy objects The most important concept in using variables is that of the *life span* or *scope* of the object it represents. During the life span of the object, the chosen letter stands for the object. At the end of its life span, the letter no longer represents that object. There are clear rules for determining the life span of an object, and these will be discussed in detail.

Objects whose life span is one sentence of less are called *dummy objects*, and the letter representing it is called a *dummy variable*, *temporary variable* or *bound variable*. Dummy objects are not part of a scenario; they are used exclusively to make it easier to express what the sentence says. It is always possible to express the meaning of the sentence without the dummy variable (although the sentence may become harder to understand.) For example,

- A varaible used with a quantifier such as "For all x, $P(x)$" or "There exists x such that $P(x)$." If we want to say "There is a number x such that $x^x = 2$", we could say it without the variable as "There is a number which when raised to the power of itself gives 2".

- A variable used in the definition of a set of a function such as: "Let S be the set $\{x \in \mathbb{R} : \sin(x) \leq 1/2\}$". In this sentence we are defining the set S and S is an object name. However, x is a dummy variable; its purpose is to make it easier to define the set S. The variable x only has a meaning within the set brackets. We could say the same sentence without the dummy variable as "Let S be the set of real numbers whose value in the sin function is at most 1/2." Another example is "Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x + x^2$." The letter x is a dummy object used to help define the primary object f.

- A variable used as an index of summation. In the expression $\sum_{i=1}^{10} i^3$, the variable i is a dummy variable. We could have expressed this without the variable i by saying "The sum of the cubes of all integers from 1 to 10"

Objects with longer life span are part of the current scenario and are called *primary objects*, and the letters representing them are called *object names*. A primary object is introduced for a specific purpose within the scenario, and that purpose determines its life span. In a proof, the life span is determined by the goal at the time the object is introduced. Once that goal is achieved, the life span of the object ends.

Often the life span of an object is an entire proof. However, there are three situations where the proof you are doing breaks up into two or more separate *subproofs*:

- When using proof by cases, each case requires a separate subproof.

- Proving "A and B" where A and B are assertions, usually requires proving A and B separately. The proof of A and the proof of B are separate subproofs.

- Sometimes in a long proof, you need to use a fact in your proof, and that fact itself needs to be proved. In this case, you interrupt the main proof and state the fact as a separate proposition (often labeled as a "Lemma") and then prove that fact. The proof of this lemma is a subproof of the main proof.

The general rule is: Any object introduced inside of a subproof dies at the end of the subproof.

The central rule for assigning variable names to objects in a mathematical discussion is:

> **Unique Names rule** Two objects with overlapping life spans must be assigned different letter names. If a letter is used as a primary object, then during the life span of the object, you can not use that letter for any other object, including a dummy object.

8.1 Primary objects

Primary objects are part of the scenario under discussion. Primary objects are either *hypothetical objects* or *auxiliary objects.* Both are mainly used in proofs, but may also appear in definitions.

- Primary objects are used to prove a universal assertion. If our current goal is to prove "For all $x \in T$, $A(x)$", we introduce a hypothetical object into the scenario "Suppose x is an arbtrary member of x", and replace the goal with "Prove $A(x)$".
- A primary object that is not a hypothetical object is an auxiliary object. Auxiliary objects are introduced in proofs to help achieve the goal of the proof.

For primary objects, the unique object names rule has the following interpretation: The name used for a primary object *can not be used for any other purpose* during the life span of that object.

The other crucial rule for primary objects is:

> **Object introduction rule** Whenever you use a letter to represent a primary object in a mathematical proof or discussion, the *very first time* that you mention that letter you must give a *proper introduction* that tells the reader what that letter will represent in your discussion.

This rule applies to both hypothetical objects and auxiliary objects, but the meaning of "proper introduction" differs for these two forms of objects.

Introducing hypothetical objects When proving a universal assertion of the form "For all x belonging to T, $A(x)$ holds"' we start the proof:

> Suppose x is an arbitrary member of T.

- The use of "Suppose" and "arbitrary" alerts the reader that x is being introduced as a hypothetical object in order to prove a universal assertion.

- In the statement of the universal assertion, x is a d] object (since it appears with a "for all" quantifier), and the life span of x is just that one sentence. To convert x into a primary object we need the introduction.

- It is not necessary to name the hypothetical object with the same letter that was used as a dummy variable in the statement of the universal assertion. We could have used any object name that was not currently in use.

- *A commonly used shortcut that is not for novice provers.* When we use the hypothetical object method to prove a universal proposition, we introduce a hypothetical object (or objects) to correspond to the universally quantified variable (or variables) in the proposition. Mathematicians often skip the explicit introduction "Suppose x is an arbitrary member of T" and simply proceed with the proof as though x had been introduced. This is an *implied introduction.* The experienced mathematician reading such a proof recognizes that there is an unwritten introduction intended. For the inexperienced proof-writer or proof-reader, the explicit introduction helps to keep clear that x has changed from a dummy variable to an object name.

Introducing auxiliary objects An auxiliary object is a primary object other than a hypothetical object that is introduced into your proof. The auxiliary object is introduced in order to help achieve the goal.

Example 8.1. Let's consider the following assertion: The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x - 1$ is onto.

Proof. By definition, f is onto means that for every $y \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = y$. Suppose y is an arbitrary real number. We must show that there is an $x \in \mathbb{R}$ such that $f(x) = y$. Since the sum of two real numbers is real, $y + 1$ is a real number, and since the quotient of a real number and 2 is real, $(y + 1)/2$ is real. Let x be the real number $(y + 1)/2$ Thus x is a real number, and $f(x) = f((y + 1)/2) = 2(y + 1)/2 - 1 = y$, as required. □

In this proof, y is a hypothetical object, and x is an auxiliary object. We normally introduce an auxiliary object using the word "Let" instead of "Suppose"; below we'll see an alternative way to introduce an auxiliary object.

Remark 8.1. In this course we are careful to distinguish hypothetical objects and auxiliary objects by using different terminology to introduce them. We do this in order to assist students in learning the difference between the different types of primary objects. Mathematicians are not careful about this, they generally use "Let" or "Suppose" to introduce either kind of object. They may also introduce an object with "Assume x is an object of type T"; we use assume only to impose an assumption about objects that are already part of the scenario.

The crucial way that auxiliary objects are treated differently from hypothetical objects is:

> (**Existence first rule**) Before introducing an auxiliary object with certain properties, you must *demonstrate that such an object exists.* This demonstration may use known facts and assumptions in the present scenario.

Thus, in the proof above, before introducing $x = (y+1)/2$ we first explained why $(y+1)/2$ is a real number. The fact that $(y+1)/2$ is a real number is very elementary and we normally won't prove something so basic. But it is not always clear that the object that you want to introduce actually exists. The following example shows why we must be sure it exists before we introduce it.

Example 8.2. Consider the following obviously false assertion. "For all real numbers r and s, $r < s$." Here is a proposed proof.

Invalid Proof: Suppose that r and s be arbitrary real numbers. Let A be the set of all real numbers that are greater than r. Let B be the set of real numbers that are less than s. Let $x \in A \cap B$. Then since $x \in A$, we have $r < x$ and since $x \in B$ we have $x < s$. Since $r < x$ and $x < s$ we must have $r < s$.

Since r and s were arbitrary, we conclude that for all all real numbers r and s, $r < s$. $\square$

The proof must be faulty, since it is proving a false statement. We can uncover the error using the method of *test objects*. Namely, we go through the proof and substitute specific objects (test objects) for the hypothetical values, and try to follow along the rest of the proof, defining the auxiliary objects as we go. If the proof is correct, we should be able to do this no matter how we choose the test objects.

In the above proof, if we try the test objects $r = 3$ and $s = 7$ then A must be the set $\{z \in \mathbb{R} : z > 3\}$ and B must be the set $\{z \in \mathbb{R} : z < 7\}$ and $A \cap B$ is the set $\{z \in \mathbb{R} : 3 < z \wedge z < 7\}$. We then choose x from $A \cap B$, for instance $x = 5$ Indeed $3 < 5$ and $5 < 7$ so $3 < 7$.

However, if we try the test objects $r = 9$ and $s = 6$ then A is the set $\{z \in \mathbb{R} : z > 9\}$ and B is the set $\{z \in \mathbb{R} : z < 6\}$ and so $A \cap B = \{z \in \mathbb{R} : z > 9 \wedge z < 6\}$. However, when we go to choose x from $A \cap B$, we see $A \cap B$ is empty, so its impossible to select x.

The Existence First rule protects against the error made in the previous proof. Under this rule, before you can introduce x to be a real number that belongs to both A and B, you would first need to prove, "There exists a real number z such that z belongs to both A and B." Only after proving this could you introduce x.

In general, if you want to introduce an auxiliary object:

> Let x stand for an object that satisfies the condition $P(x)$,

then you need to first prove: "there is at least one object z that satisfies $P(z)$."

In a previous example, which gave a correct proof that the function f is onto. We introduced x to be $(y+1)/2$. Before doing this we explained why $(y+1)/2$ is a real number. In this case, the fact that $(y+1)/2$ is a real number is so basic that we could probably omit it. So we might shorten the proof to:

Suppose y is an arbitrary real number. We must show that there is a real number x such that $f(x) = y$. Let $x = (y+1)/2$ Then $f(x) = f((y+1)/2) = 2(y+1)/2 - 1 = y$, as required.

This is fine, but slightly dangerous. Suppose $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ is given by $g(x) = 2x - 1$ and we tried to prove that g is onto. We could try using the same proof, replacing f by g and "real number" by integer The problem is that x as defined is not an integer if y is even. So the definition of x is valid but it is not an integer, and so does not meet the requirements.

An alternative way to introduce auxiliary objects We've seen that we introduce hypothetical objects with a sentence starting "Suppose" and auxiliary objects with a sentence starting "Let". There is another way to introduce auxiliary objects. Let's consider an example.

Proposition 8.1. *Every composite integer n has a divisor that is greater than 1 and less than or equal to $\sqrt{n}$.*

Proof. Suppose n is a composite integer. We must show that there is a divisor of n that is greater than 1, less than or equal to $\sqrt{n}$. By the definition of composite, there exists a number k such that k divides n and $k > 1$ and $k < n$. Let k be a number satisfying these properties. We consider two cases, depending on whether $k \leq sqrtn$ or $k > \sqrt{n}$.
Case 1. Assume $k \leq \sqrt{n}$. Since k is a divisor of n and $k > 1$, k satisfies the requirements of the proposition.
Case 2. Assume $k > \sqrt{n}$. Let $a = n/k$. Since k is a divisor of n, then a is an integer. Since $k < n$, we have $a > 1$. Since $k > \sqrt{n}$ we have $a = n/k < n/\sqrt{n}- = \sqrt{n}$ ☐

There are two auxiliary variables in this proof, k and a. Look at the introduction of lk. We have two sentences:

> By the definition of composite, there exists a number k such that k divides n and $k > 1$ and $k < n$. Let k be a number satisfying these properties.

The second sentence seems a repetitive. It looks like the first sentence introduces k. Why do we need the second sentence? The reason is technical: In the first sentence, k is a dummy variable (since it is attached to an existential quantifier. This sentence says that there is a number k with certain properties, but k has not been introduced into the scenario and can not be referred to in later sentences. Hence we use the second sentence to propertly introduce k into the scenario.

Now, it is common practice among mathematicians to skip the second sentence, and to treat the first sentence as though k were actually being introduced as an auxiliary variable. While you may do this eventually, for now we want to avoid this improper usage until you've mastered the difference between dummy object, hypothetical object and auxiliary object.

So there is an alternative way to introduce auxiliary objects:

> By the definition of composite, there exists a number which we call k such that k divides n and $k > 1$ and $k < n$.

Here we took the first sentence and added the phrase "which we call" before k. This phrase serves to announce that we are introducing k as an auxiliary object, and not just using k as a dummy variable.

The reason we are careful about the distinction of dummy variables and auxiliary variables is illustrated by the following example. For this example we use the following definition of what it means that k is a divisor of n: k is a divisor of n if there is an integer s such that $ks = n$. (This is equivalent to the other definition we have used: k is a divisor of n if n/k is an integer.)

Proposition 8.2. *For any two integers n and m, if k is a divisor of n and k is a divisor of m then k is a divisor of $m + n$.*

Invalid Proof: Suppose n and m are arbitrary integers. Assume k is a divisor of n and k is a divisor of m. Since k is a divisor of n there is an integer s such that $n = sk$. Since k is a divisor of m there is an integer s such that $m = sk$. Therefore $m + n = sk + sk = 2sk$. Since $2s$ is an integer k is a divisor of $m + n$. □

To see what's wrong with this proof, let's test it with $n = 12$ and $m = 15$ and $k = 3$.

In the sentence "Since k is a divisor of n there is an integer s such that $n = sk$" s is a dummy variable. The s that works is 4. In the sentence "Since k is a divisor of m there is an integer s such that $m = sk$", s is a again a dummy variable, and the s that works is 5. Neither sentence introduces s into the scenario. Our next sentence says $m + n = 2sk$. In our example $m + n$ is 27. What is s? Niether 4 nor 5 works.

In fact, if this proof were correct it seems to prove something stronger: if k is a divisor of n and k is a divisor of m then $2k$ is a divisor of $m + n$. This is simply not true!

The way to avoid this problem is to carefully apply the rules for using variables. When we first mention s we should introduce s:

"Since k is a divisor of n there is an integer which we will call s such that $n = sk$" Now that s is introduced we know we can't use the letter s for any other purpose. In the next sentence we need to replace s by another letter: "Since k is a divisor m there is an integer which we will call t such that $n = tk$." Now s and t are auxiiliary variables that refer to different integers. The full correct proof becomes:

Proof. Suppose n and m are arbitrary integers. Assume k is a divisor of n and k is a divisor of m. Since k is a divisor of n there is an integer which we'll call s such that $n = sk$. Since k is a divisor of m there is an integer which we'll call t such that $m = tk$. Therefore $m + n = sk + tk = (s + t)k$. Since $s + t$ is an integer k is a divisor of $m + n$. □

A common mistake. Adding the phrase "which we call" as above allows us to combine a sentence that states the existence of an object, with the introduction of the object into the scenario. We may only do this if the existential statement is something we have actually proved. Here's an alternate invalid proof of the previous proposition.

Invalid Proof: Suppose n and m are arbitrary integers. Assume k is a divisor of n and k is a divisor of m. We must show that there is an integer which we'll call s such that $sk = m + n$. Since s is an integer k is a divisor of $m + n$, as required. □

What happened in this proof? The sentence "We must show " announces our goal is a to find an integer that when multiplied by k is $m + n$. We haven't found the integer yet! It is invalid to introduce s into our scenario because we don't yet know it exists.

We have said that when you introduce an auxiliary object with certain properties, we must first prove that it exists. In many cases this proof is so obvious that we skip it.

Example 8.3. Suppose x is a real number in our scenario and we want to introduce y to be $2x + 1$. Since we know that 2 times a real number is a real number, and adding 1 keeps it a real number, we simply write "Let $y = 2x + 1$." We don't need to prove that y exists.

Example 8.4. Suppose x is a real number in our scenario and we want to introduce y to be $1/(x-1)$. Now we have to be careful. If $x = 1$ then y does not exist. So we have to either (1) prove that in our current scenario x can not equal 1, or (2) break the proof into cases where one case is $x \neq 1$ where we can introduce y as above, and the other case $x = 1$ has to be handled by another method.

Example. Suppose we are trying to prove: "For all integers n, $n^2 + n$ is even."

Here is an invalid proof.

> Suppose n is an arbitrary integer. Let k be the integer $(n^2 + n)/2$. We have that $n^2 + n = 2k$ which is twice an integer so n^+n is even.

The problem comes in the second sentence. We can introduce $k = (n^2 + n)/2$, where k is a real number, but we are claiming that k is an integer, and this needs to be proved.

Remark 8.2. (This remark is optional reading.) Why it isn't necessary to prove "existence first" for hypothetical objects. The Existence First rule applies only to auxiliary objects. Why doesn't it apply to hypothetical objects?

The short answer is: failure to follow the Existence First rule for auxiliary objects can lead to proving a false assertion. There is no such risk for hypothetical objects.

Suppose we are trying to prove a universal proposition of the form "For all x satisfying $H(x)$, $C(x)$ must hold." We start the proof "Suppose x is an object satisfying $H(x)$." We then complete the proof by showing that $C(x)$ must also be true.

If we succeed, we declare that the proposition must be true. But if we did not first establish that an x satisfying $H(x)$ exists, how can we really be sure that the proposition is true?

There are two possibilities

- There really is at least one x satisfying $H(x)$. Then our proof shows that any such x must satisfy $C(x)$, so the proposition is true.
- What if there is no x satisfying $H(x)$? Then the proposition is still true, because it is *vacuously true.*

So if our proof works, we will correctly conclude that the proposition is true.

Let's consider an example:

Example. For all real numbers x satisfying $x = x + 1$, we have $x = x + 2$.

Of course there are no x's satisfying the hypothesis. If we don't realize this we might write:

> Let x be a real number satisfying $x = x + 1$. Adding 1 to both sides we have $x + 1 = x + 2$. Then $x = x + 1 = x + 2$.
>
> Since x was arbitrary, we conclude that for all real numbers x satisfying $x = x + 1$, we have $x = x + 2$.

Since we succeeded in proving the proposition, we say that it's true. And, indeed, it is (vacuously) true!

Remark 8.3. Some other commonly used object introductions Mathematicians sometimes use other language to introduce primary objects such as "Consider an integer a" or "Assume a is an integer".

When you read mathematics, you may see these and other introductions. For now, until you become accustomed to proper introduction of object names, it is strongly suggested that you not use these and that you use the above construction with "Let ...".

Also mathematicians are generally not so careful about using the word "arbitrary". They may omit it when introducing a hypothetical object, or include it when introducing an auxiliary object.

8.2 The "Complete specification" (CS) rule

This rule applies to both hypothetical object names and auxiliary object names. When you introduce an object and give it a name, you must completely specify all the properties that you are assuming about the object.

For example, here's a simple example involving the introduction of a hypothetical object:

> For all real numbers x satisfying $x \geq 1$, $x^2 \geq x$.

The proof starts:

> Suppose x is an arbitrary real number satisfying $x \geq 1$.

When introducing a hypothetical object, it is permissible to use two sentences. For example:

> Suppose x is an arbitrary real number. Assume $x \geq 1$.

The first sentence introduces x but does not specify all of the hypotheses. The second sentence completes the specification. This second sentence starts with "Assume ...".

The word "Assume" in proofs is often misused by students. Here, we are using "Assume" properly, to complete the specification of a hypothetical object. The other situations where we can add assumptions in a proof were discussed in Section 7 There are a three other situations where we can add assumptions in a proof: when our current goal is an "if-then" assertion we assume the "if" part and our new goal is the "then" part, or when we use proof by cases, each case starts with an assumption.

The word "assume" is important. In the previous example, it would be improper simply to write

> Let x be an arbitrary real number. Then $x \geq 1$.

This is incorrect because here the writer is claiming that "$x \geq 1$" must be true no matter what real number x is. Of course, this is not true.

Example. Suppose you are proving a universal-existential proposition such as:

Proposition 8.3. *For any two real numbers x and y satisfying $x < y$, there is a real number w satisfying $x < w$ and $w < y$.*

Proof. Suppose x and y are arbitrary real numbers satisfying $x < y$. Let $w = (x + y)/2$. We now show $x < w$ and $w < t$. Since $x < y$ we have $x+x < x+y$ so $x = (x+x)/2 < (x+y)/2 = w$. Similarly $x < y$ implies $x + y < y + y$ so $w = (x + y)/2 < (y + y)/2 = y$. Thus $x < w$ and $w < y$ as required. □

Here is a similar but *incorrect* way of writing the proof.

> Suppose x, y and w are arbitrary real numbers satisfying $x < y$. Let $w = (x+y)/2$.

This is incorrect for two reasons:

- The initial sentence in a proof by the hypothetical object method should only introduce the hypothetical objects, which correspond to the objects in the proposition that are universally quantified. The proposition says "For all x and y [something is true]" where the "something" involves the existence of a real number w. It does not say "For all x, y and w [something is true]", so w should not be introduced in the initial sentence.
- You should not introduce the same object twice, but w is introduced in the first sentence, and then reintroduced in the second.

8.3 Dummy objects

We now turn to a brief discussion of the rules involving djummy objects. A dummy object is an object whose scope (life span) is one sentence or less. A letter representing a dummy object is called a *dummy variable*, *temporary variable* or *bound variable.*

There are two important usage rules for dummy variables:

Dummy objects are temporary

Unique names for dummy objects

8.4 Dummy Variables are Temporary

The life span of a dummy variable is one sentence or less. When you use a letter as a dummy variable, this is very different than introducing it as an object name. After the sentence in which the dummy variable is used, you can not refer to the letter as an object.

Example 8.5. Here is a common error:

> Let $A = \{x \in \mathbb{R} : 0 < x < 1\}$. Then $x < 2$.

The first sentence defines the object A using x as a dummy variable. The variable x was not introduced and can not be used.

Here is something similar that is permitted.

Let $a = \{x \in \mathbb{R} : 0 < x < 1\}$. Then for all $x \in A$, $x < 2$.

Again x is a dummy variable in the first sentence. In the second sentence we reuse x as a dummy variable in the second sentence. We are allowed to reuse the variable x in the second sentence because the lifespan of the first usage of x ends outside of the set brackets.

In the previous example, x is not a dummy variable in the sentence "Then $x < 2$" because it we take that sentence by itself, x is a free variable and not bound.

Unique names for dummy objects This rule is simple, and follows from our general principle of not using a letter for two different purposes at the same time: While you are using a particular letter as an object name, don't also use it as a dummy variable.

Here is a common violation of this rule.

Suppose x and y are arbitrary real numbers. For all $x, y \in \mathbb{R}$, $x^2 - y^2 = (x+y)(x-y)$.

In the first sentence you have introduced x and y to be real number objects. In the second sentence, you use x and y as dummy variables. The following would be acceptable.

Suppose x and y are arbitrary real numbers. Since for all real numbers a and b we have $a^2 - b^2 = (a + b)(a - b)$, we must have $x^2 - y^2 = (x + y)(x - y)$.

or,

Suppose x and y are real numbers. By elementary algebra, $x^2 - y^2 = (x-y))(x+y)$.

Reusing dummy variables Since dummy variables have short life spans, they are often reused. For example, the following is permissible:

Let $\hat{A} = \{x \in \mathbb{R} : x^2 + x \geq 12\}$. Let $\hat{B} = \{x \in \mathbb{R} : x^2 + x \geq 15\}$.

In this case, $\hat{A}$ and $\hat{B}$ are both object names. x is a dummy variable used to define $\hat{A}$. At the end of the definition of $\hat{A}$, the dummy variable x "dies", and so can be reused.

The following is also allowed:

For all real numbers x that are greater than 1, $x^2 > x$ and for all real numbers x that are greater than 2, $x^2 > 2x$.

Here the same dummy variable is used in two different ways in the same sentence, but the uses are clearly separated because of the structure of the sentence as an "and" (or *conjunction*) of two propositions.

Here are sentences involving dummy variables that don't make sense and are therefore not permitted.

For all x, there exists x, such that $x^2 \geq x$.

Let $A = \{n \in \mathbf{N} : \exists n \text{ such that } n \text{ is even}\}$.