5 Tools from logic [6]

Our goal is to be able to tell which mathematical assertions are true and which are false. The field of *mathematical logic* is concerned with developing a general methodology for studying mathematical assertions that works for all mathematical assertions. This turns out to be a very difficult subject, with many subtle and complicated issues.

Fortunately, the majority of problems that mathematicians study do not require deep understanding of these logical issues, and mathematicians can function quite effectively with only a very basic knowledge of mathematical logic. Section 3 covered some of this basic knowledge: definite and indefinite assertions, and universal and existential quantification. In this section we cover some additional background in mathematical logic.

- Construction of more complex assertions by combining or modifying simpler assertions
- Ways to compare two assertions: Basic logic allows us to say that one assertion is a *logical consequence* of another, so that if we know the second assertion is true then we can conclude that the first is true. Also, two assertions are *logically equivalent* if each is a logical consequence of the other.

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5.1 Building new assertions from old

There are various ways to modify and combine assertions to create new assertions. In the previous section we introduced *quantification*, which converts an assertion with a free variable into an existential or universal assertion. In this section we'll introduce some other important ways to change an assertion: *negation*, and *logical combination*.

The negation of an assertion Any assertion makes a specific claim that is either true or false (which for indefinite assertions may depend on the values of some variables.) Given an assertion A, the assertion "It is not the case that A" is called the *negation of the assertion* A, and is abbreviated $\neg A$.

For example:

4. The negation of the definite assertion "$59 \times 48 \geq 52^2$" is the sentence "It is not the case that $59 \times 48 \geq 52^2$" or "$59 \times 48 \geq 52^2$ is false"'
5. The negation of the definition assertion "Every even integer bigger than 2 can be expressed as the sum of two primes." is the sentence "It is not the case that every even integer bigger than 2 can be expressed as the sum of two primes"
6. The negation of the indefinite assertion "$x^3 + x \geq 5x^2$" is "It is not the case that $x^3 + x \geq 5x^2$".

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The phrases "It is not the case that" and "is false" are *negating phrases* that convert a sentence to its negative. It is often useful to be able to reformulate the negated sentence without negating phrases. For example, Assertion 4 means the same as "$59 \times 48 < 52^2$". For Assertion 5 we can rewrite the negated sentence as "There is an even integer bigger than 2 that can't be expressed as the sum of two primes."

For an assertion A, $\neg A$ is true exactly when A is true and $\neg A$ is false when A is true.

Combining two assertions Suppose that A and B are assertions. There are four basic ways to combine A and B into a new assertion:

and ($\wedge$) The assertion "A and B" is true provided that both A and B are true. The symbolic abbreviation is $A \wedge B$.

or ($\vee$) The assertion "A or B" is true provided that at least one of A and B is true. The symbolic abbrevation is $A \vee B$.

implies, if-then ($\implies$) The assertion "A implies B", also written "if A then B" means that "if it is the case that A is true then B must also be true". This sentence is considered false if A is true and B is false, and is considered true otherwise. In particular, if A is false then $A \implies B$ is true, whether or not B is false. The symbolic abbreviation of "A implies B" is $A \implies B$.

if and only if ($\iff$) The assertion "A if and only if B", is true provided A and B are both true or A and B are both false. The symbolic abbreviation is $A \iff B$.

When we combine two assertions using one of these connectives, the truth value of the resulting assertion depends only on the connective used, and the truth value of the assertions being combined. The following table summarizes this:

$TV(A)$	$TV(B)$	$\wedge$	$\vee$	$\implies$	$\iff$
T	T	$T \wedge T = T$	$T \vee T = T$	$T \implies T = T$	$T \iff T = T$
T	F	$T \wedge F = F$	$T \vee F = T$	$T \implies F = F$	$T \iff F = F$
F	T	$F \wedge T = F$	$F \vee T = T$	$F \implies T = T$	$F \iff T = F$
T	T	$F \wedge F = F$	$F \vee F = F$	$F \implies F = T$	$F \iff F = T$

You can combine any two assertions by any of these methods. For example, from the two assertions: "7 is prime" and "13 is divisble by 4" we can build the following assertions:

7. 7 is prime and 13 is divisible by 4.
8. 7 is prime or 13 is divisible by 4.
9. if 7 is prime then 13 is divisible by 4.
10. if 13 is divisible by 4 then 7 is prime.
11. 7 is prime if and only if 13 is divisible by 4.

Since “7 is prime” is true and “13 is divisible by 4” is false, Assertion 7 is false, Assertion 8 is true, Assertion 9 is false and and Assertion 11 is false.

Remark 5.1. The mathematical use of “and”, “or”, “implies” and “if and only if” is similar to their use in everyday language, but there are some differences.

1. When someone says “John and I are going to the movie” then “and” does not connect two assertions, it connects two nouns “John” and “I”. This sentence is a shortened way to write “John is going to the movie and I am going to the movie”, which is the combination of two assertions with “and”. A mathematician might write “5 and 7 are prime” or “5 or 6 is even” but it is important to realize that logically the first is shortened way to write “5 is prime and 7 is prime”, and the second is a shortened way to write “5 is even or 6 is even”.

2. In mathematics, “A or B” means that at least one of A and B is true, and possibly both are true. In everyday language, “A or B” might have a similar meaning, or it might mean that exactly one of A and B is true, for example, “I will have pizza for dinner or I will have sushi for dinner” usually means “I will have pizza for dinner or I will have sushi for dinner, but not both”. When a mathematician who means “A or B but not both” must clearly say this or something like it, such as “Exactly one of A and B is true”.

3. In a statement “if A then B”, the assertion A is called the *assumption* and B is called the *conclusion* and the meaning is “if A is true then B is also true”. The only way the sentence is considered false is if A is true and B is false. In everyday usage, when we say “if A then B” or “A implies B” we normally use it in a situation where A can be thought of as *causing* B. We might say, “if I miss the bus then I will be late for my appointment” Here the first part is the assertion “I miss the bus” while the second assertion is “I will be late for my appointment” and the first assertion is the cause of the second. In mathematics,“A implies B” is usually used whtn A and B are related, but the rules of logic don’t require this, and “A implies B” even when A and B have no connection. Such a sentence is true unless A is true and B is false, in which case it is false.

Remark 5.2. **Combining indefinite assertions.** Indefinite assertions can be combined in the same way that definite assertions are. However, there are a few things to be aware of.

- Don’t use the same letter as both a dummy variable and a free variable in a single sentence. Suppose we have the sentences “$x \leq 7$” and “For all x, $x^2 + 1 \geq x$. In the first sentence x is a free variable, while in the second it isi a dummy variable. If you combine these sentences, using “and” for instance, you get “$x \geq 7$ and for all x, $x^2 + 1 \geq x$. The use of x as both a free variable and a dummy variable in the same sentence is potentially confusing, and should not be done. Here’s how to avoid it. Recall that in an assertion where x is a dummy variable we may replace all occurences of x by another letter. So replace all x’s in the second sentence by a different letter, say z. The combined sentence will then be “$x \geq 7$” and for all z. $z^2 + 1 \geq z$.

- Consider the two sentences "If n is prime and $n > 2$ then n is odd." and "For all n, if n is prime and $n > 2$, then n is odd". From a mathematical standpoint these sentences are different. In the first sentence n is a a free variabe. The second assertion is a definite assertion with dummary variable n. The second sentence is true. Since the first sentence is an indefinite assertion it does not have a truth value, but if we substitute a specific value for n it does have a truth value.

  Even though the first sentence is different from the second, it is common, even among mathematicians, to treat the first sentence as though it has "For all n" added to the beginning so that it means the same as the second sentence. This is an example of a violation of the "safety rules" of mathematical communication. It happens to be a violation that is not that dangerous (it's unlikley to cause confusion) but it is better for students in this course not to violate the safety rule and to treat these sentences as different.

Logical expressions. In elementary algebra, you dealt *algebraic expressions* such as

$$5 \times a \times (b + c \times ((a - b) - 4) + 7 \times d.$$

These expressions are built by starting with variables, each representing a real number, and combining them with each other, or with constants by using addition, multiplication and subtraction.

In logic, one deals with *logical expressions*. In such expressions. the variables, called *propositional variables* represent assertions, which are combined or modified using $\neg$, $\wedge$, $\vee$, $\implies$ and $\iff$, $\forall$ and $\exists$.

Here's an example of a logical expression:

$$\exists x, A(x) \wedge \forall z, B(z) \implies C(z).$$

In the logical expression above, A, B and C represent assertions. We write $A(x)$ instead of A to eimphasize that x appears as a free variable in A, and similarly z appears as a free variable in B and C

Modeling assertions by logical expressions Logical expressions allow us to study the way assertions are put together from certain starting assertions, without worrying what the starting assertions are. In the above example the expression is built from starting assertions A, B and C.

Example 5.1. Suppose we are studying the assertion "For every integer n, if n is prime and $n - 1$ is divisible by 4, then there are integers a and b such that $n = a^2 + b^2$. We can represent this by logical expressions in various ways, depending on how we assign our variables:

- We could simply assign a single variable A to the entire assertion, and our logical representation is simply A.

- We could use $B(n)$ to represent the indefinite assertion "if n is prime and $n-1$ is divisible by 4 then there are integers a and b such that $n = a^2 + b^2$" and then our entire sentence is represented by "$\forall n, B(n)$".
- We could let $C(n)$ be the indefinite assertion "n is prime", let $D(n)$ be the indefinite assertion "$n - 1$ is divisible by 4", and $E(n)$ be the indefinite assertion (with dummy variables a, b) "There are integers a and b such that $n = a^2 + b^2$". In this case our sentence will have the representation "$\forall n, (C(n) \wedge D(n)) \implies E(n)$".
- We can let $F(n, a, b)$ denote the indefinite assertion with free variables a, b, n given by "$n = a^2 + b^2$". We can then rewrite our sentence as $\forall n, (C(n) \wedge D(n)) \implies \exists a \exists b F(n, a, b)$.

All of these logical expressions represent the original assertion; the later ones provide more detail on the structure of the assertion by breaking it down further.

An assertion is called *atomic* if it does not use an existential quantifier "there exists x" or a universal quantifier "for all x" or the connectives "and", "or", "implies (if-then)" or "if and only if". In the previous example all of the assertions $C(n)$, $D(n)$ and $F(n, a, b)$ are atomic expresions. Here are other examples of atomic assertions: "$x^2 \neq 7$" or "$f(x) \leq f(z)$, and "$S \subseteq T$" Note that if A is atomic, then we also consider its negation $\neg(A)$ to be atomic.

Remark 5.3. The assertion "$A \subseteq B$" is atomic, but if we use the definition of "$\subseteq$", we can rewrite the sentence as: "for all $x \in B$, we have $x \in A$". This is not an atomic assertion since it has a universal quantifier. Thus it's possible that two sentences can have the same meaning where one is atomic and the other iis not.

An assertion that is not atomic is called a *compound assertion*. Every compound assertion is of one of the following forms:

decomposable An assertion of the form "A and B", "A or B", "A implies B" or "A if and only if B", where A and B are assertions, is said to be *decomposable*, because it can be decomposed into two assertiions. Note that A and B might themselves be compound assertions.

quantified An assertion of the form "for all x, $A(x)$" or "there exists x such that $A(x)$" is a *quantified assertion*. Here $A(x)$ might itself be a compound assertion.

negative compound assertion An assertion of the form "it is not true that A" where A is a compound assertion. is a negative compound assertion.

Representation of compound assertions using a logical expression will be very useful to us when we start doing proofs. The following three concepts will be especially important.

- The *top-level* structure of a compound assertion
- *Logical equivalence* of two compound assertions
- One compound assertion is a *logical consequence* of another.

Top-level structure of a compound assertion Every compound assertion can be written in exactly one of the following 7 forms, by specifying the assertions A and/or B appropriately. For a given compound assertion, the form that fits it is called the *top-level structure* of the assertion.

$A \wedge B$

$A \vee B$

$A \implies B$

$A \iff B$

$\neg A$

$\exists x, A$

$\forall x, A.$

Example 5.2. The sentence "For all x, $f(x) \leq f(y)$ implies $x \leq y$, and for all z, $z \leq f(z)$". This assertion has two free-variables f (standing for a function) and y standing for a real number. The form of the sentence is $(\forall x, (C(x, y, f) \wedge D(x, y))) \wedge (\forall z, E(z, f))$ where $C(x, y, f)$ is the assertion "$f(x) \leq f(y)$" and $D(x, y)$ is the assertion "$x \leq y$" and $E(z, f)$ is the assertion"$z \leq f(z)$". Notice that the way this assertion is put together: $C(x, y, f)$ is combined with $D(x, y)$ using $\wedge$ and then the combined assertion is given a universal quantifier in x, to produce an assertion $G(y, f)$. Separately, $E(z, f)$ is given a universal quantifier in z to produce an assertion $H(f)$. The entire assertion is then constructed as $G(y, f) \wedge H(f)$, and this is the top-level structure.

Remark 5.4. There is some possible ambiguity in the meaning of the original assertion because we need to know how the parts of the sentence are grouped. A different interpretation would be "For all x, $(C(x, y, f) \wedge D(x, y) \wedge \forall z, E(z, f))$. In this interpretation, the top-level structure is $\exists x, J(x)$, where $J(x)$ represents the inner assertion above. How do we know which is meant? There are subtle reasons to prefer the first interpretation, but the possible ambiguity means that the sentence as written is potentially dangerous! (Any sentence that does not have a single clear interpretation is dangerous for mathematical communication.) The way we can get around this is to use parentheses in complex English sentences as they are used in mathematical sentences. So to be safe, we might write "(For all x, $f(x) \leq f(y)$ implies $x \leq y$) and (for all z, $z \leq f(z)$).

Example 5.3. The assertion "$x^2 \geq x$ implies that there is a number z such that $z^2 + 1 \leq 2xz$ and $z \geq 1$" has top-level structure $A(x) \implies B(x)$ where $A(x)$ is the sentence "$x^2 \geq x$" and $B(x)$ is the sentence " there is a number z such that $z^2 + 1 \leq 2xz$ and $z \geq 1$."

Since $B(x)$ is a compound assertion, we can take our analysis further to determine the top-level structure of $B(x)$. Here $B(x)$ has top-level structure $\exists z C(x, z)$ where $C(x, z)$ is the assertion "$z^2 + 1 \leq 2xz$" and $z \geq 1$". Also, $C(x, z)$ has top-level structure $D(x, z) \vee E(z)$, where $D(x, z)$ is the assertion "$z^2 + 1 \leq 2xz$" and $E(z)$ is the assertion "$z \geq 1$".

Example 5.4. Consider the assertion "It is not the case that both n is prime and $n+1$ is a square." (Note: A number is said to be a *square* if it is the square of some integer.) The top-level structure is $\neg C$ where $C = A(n) \wedge B(n))$, and $A(n)$ is "n is prime" and $B(n)$ is "$n+1$ is a square".

Logical equivalence of logical expressions and of assertions Let's start with an example. Consider the expressions "$P = \neg(A \wedge B)$" and $Q = \neg A \vee \neg B$. As usual A and B represent any assertions. These two expressions have the following remarkable property. If you replace each of the assertions A and B in both expressions by any assertion you choose, the resulting two assertions are both true or both false. Since for every choice of A and B the two expressions have the same truth value, we say that P and Q are *logically equivalent*, written $P \equiv Q$.

To see the equivalence, note that the first expression is true if $A \wedge B$ is false, and for this we need A to be false or B to be false which happens exactly when $\neg A \vee \neg B$ is true.

Two assertions V and W are *logically equivalent* if we can label the assertions appearing in V and in W by propositional variables and represent V and W by logical expressions involving those variables, in such a way that the resulting logical expressions are equivalent.

Example 5.5. Consider the two assertions in the free variable S which represents a set: "It's not true that both S is finite and S is nonempty" and "S is infinite or S is empty." By letting A be the assertion "S is infinite" and letting B be the assertion "S is empty", we can represent the first assertion by the logical expressions"$\neg(A \wedge B)$ and the second by "$\neg A \vee \neg B$". As we noted above, these logical expressions are equivalent, so the two assertions are equivalent.

Below we'll discuss ways to convert an assertion into a logically equivalent assertion.

The "logical consequence" relationship for expressions and assertions Consider the two logical expressions "$P = A \iff B$" and "$Q = A \wedge\ /B$". These expressions are not logically equivalent: If A is a true assertion and B is a false assertion than P is false, while Q is true. Nevertheless there is a weaker relationship between the expressions: anytime that P is true, Q is too. We say that Q is a *logical consequence* of P, denoted $Q \dashv P$. To see this relationship consider each of the possible pairs of truth values for A and B (both true, A true and B false, A false and B true, both false) and check which of these make P true and which make Q true. You'll see that every choice that makes P true also makes Q true.

Now if V and W are assertions, we say that V is a logical consequence of W written $V \dashv W$, provided that we can label the assertions that make up V and W by propositional variables and represent V by a logical expression Q and W by a logical expression P, involving these variables, so that $Q \dashv P$.

Example 5.6. Suppose that V and W are the assertions $V =$"$f(x) \geq y$ or it is not the case that $g(y) \leq x$" and $W =$"$f(x) \geq y$ if and only if $g(y) \leq x$". (These have four free variables, f and g representing functions and x and y representing real numbers.) Define the assertions $A =$"$f(x) \geq y$" and W is the assertion $B =$"$g(y) \leq x$." Then V is represented by the logical expression $A \vee \neg B$ and W is represented by the logical expressions $A \iff B$. Since the first logical expression is a logical consequence of the second, we conclude that V is a logical consequence of W.

Some basic logical equivalences and logical consequences To make use of logical equivalence and logical consequence, we collecgt some basis examles of pairs of logical expressions where one expression in the pair is a logical consequence or is logically equivalent to the other.

We'll start with equivalence of expressions that don't involve quantifiers. Here are some simple and important equivalences.

$$\begin{aligned}
A \vee B &\equiv \neg A \implies B \\
A \implies B &\equiv \neg B \implies \neg A \\
A \iff B &\equiv (A \implies B) \wedge (B \implies A) \\
(A \vee B) \wedge C &\equiv (A \wedge C) \vee (B \wedge C) \\
(A \wedge B) \vee C &\equiv (A \vee C) \wedge (B \vee C) \\
A \implies (B \implies C) &\equiv (A \wedge B) \implies C \\
\neg(A \wedge B) &\equiv \neg A \vee \neg B \\
\neg(A \vee B) &\equiv \neg A \wedge \neg B \\
\neg(A \implies B) &\equiv A \wedge \neg B \\
\neg(\neg A) &\equiv A.
\end{aligned}$$

You can check that these equivalences are correct by replacing each variable by **T** or **F** in all possible ways and checking that in every case, the truth value of the expression on the left matches the truth value of the expression on the right.

A typical application of an equivalence is: Suppose we are trying to prove an assertion of the form $A \iff B$. Since this is equivalent to $(A \implies B) \wedge (B \implies A)$, this tells us that in order prove $A \iff B$, we can separately prove $A \implies B$ and $B \implies A$.

Next let's consider logical consequences. Every logical equivalence gives us two consequences since $P \equiv Q$ tells us that P is a logical consequence of Q and Q is a logical consequence of P. Here are a few logical consequences that are not logical equivalences.

$$\begin{aligned}
A \vee B &\dashv A \\
A &\dashv A \wedge B \\
A &\dashv B \wedge (B \implies A) \\
A \implies C &\dashv (B \implies C) \wedge (A \implies B)
\end{aligned}$$

To see why the first logical consequence is correct, notice that when A is true it is certainly the case that at least one of A and B is true. For the second logical consequence, when both A and B are true, then certainly A is true. For the third, if we know that B is true and $B \implies A$ then it follows that A is true.

Next we'll consider logical equivalence and consequence for expressions involving quantifiers.

- $\neg\forall x, B(x) \equiv \exists x, \neg B(x)$.
- $\neg\exists x B(x) \equiv \forall x, \neg B(x)$.
- $\forall x \forall y, A(x,y) \equiv \forall y \forall x, A(x,y)$. (We can interchange two universal quantifiers.)
- $\exists x \exists y, A(x,y) \equiv \exists y \exists x, A(x,y)$. (We can interchange two existential quantifiers.)
- Suppose that $C(x)$ involves the free variable x and D does not involve x. Then $\forall x(C(x) \vee D) \equiv (\forall x C(x)) \vee D$. More generally, The same equivalence holds if we repace "$\vee$" in both formulas by any of the connectors $\wedge$, $\implies$ or $\iff$ or if we replace the quantifier $\forall$ in both places by $\exists$.
- $\forall x \exists y, A(x,y) \dashv \exists y, \forall x, A(x,y)$.

The first two equivalences, which involve negations of universal and existential assertions are the most important: when we negate a universal assertion we get an existential statement, and vice versa. To understand why this is true, note that when we say "It is not the case that for all x $A(x)$ holds" we mean "There is at least one x such that $A(x)$ does not hold" and similarly when we say "It is not the case that there is an x such that $A(x)$ holds" we're saying that "For every x, $A(x)$ doesn't hold."

For the final logical consequence, it is important to understand that these expressions are not logically equivalent. The first says that given any x, it is possible to choose y (possibly depending on x) so that $A(x,y)$ is true. The second says that it is possible to choose a single y (independent of x) so that $A(x,y)$ is true for every x. These are not equivalent, but if the second is true then the first must be true also.

Reformulations of a universal assertion The same assertion can be reformulated in many different ways. Sometimes the reformulation is based on modifying words using the fact that in English there are different ways to say the same thing. Other times, the reformulation makes use of logical equivalences. Here we'll see how this works in a simple example.

- For every positive integer n if n is prime and $n > 2$ then n is odd.

This hypothesis of the universal assertion is the scenario:
Input. A positive integer n Assumption: n is prime and $n > 2$
and the conclusion is n is odd.

Here are different ways to express this universal assertion:

1. Every prime number greater than 2 is odd.
2. Each prime number greater than 2 is odd.
3. For all prime integers n that are greater than 2, n is odd.
4. For all prime integers n satisfying $n > 2$, n is odd.

5. For all integers n such that n is prime and $n > 2$, n is odd.

6. For all prime integers n, if $n > 2$ then n is odd.

7. For all integers n, if n is prime and $n > 2$ then n is odd.

8. There are no even primes larger than 2. For all integers n, if n is even and $n > 2$ then n is not prime.

Look at each one of these carefully. All of them express the same universal proposition in different language. The words "every" and "for all" clue you in that this is a universal proposition. The words "satisfying" and "such that" after the "For all n" are clues that what follows is part of the assumption.

Assertion 8 does not start with "Every" or "For all", the statement starts "There are no ...". This is expressing the negation of the existential assertion "There is an even prime number larger than 2". We saw already that every universal assertion is the negation of an existential assertion.

Assertion 8 has the form of a universal assertion, but it seems to scramble the assumption and the conclusion. We can show that it means the same as the original assertion by using the logical equivalence rules presented earlier in this section.

The assertion we're considering is built from 3 atomic assertions: $A =$"n is prime", $B =$"$n > 2$" and $C =$"n is odd". The symbolic representation of Assertion 1 is "$\forall n, (A \wedge B) \implies C$", and the symbolic representation of Assertion 8 is "$\forall n, (\neg C \wedge B) \implies \neg A$". The fact that these two mean the same thing follows by showing that the two propositional expressions $(A \wedge B) \implies C$ and $(\neg C \wedge B) \implies \neg A$ are logically equivalent. For this we use the equivalence rules of Section 5.1.

$$\begin{aligned}(A \wedge B) \implies C &\equiv A \implies (B \implies C)\\ &\equiv \neg(B \implies C) \implies \neg A\\ &\equiv \neg(C \vee \neg B) \implies \neg A\\ &\equiv (\neg C \wedge B) \implies \neg A.\end{aligned}$$